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Intuitionistic fuzzy theory and its application in economy, technology and management

THE SPHERICAL DISTANCE FOR INTUITIONISTIC FUZZY SETS AND ITS APPLICATION IN DECISION ANALYSIS

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Abstract. Different from traditional distances between Intuitionistic Fuzzy Sets (IFS), the spherical distance between two IFSs relies not only on their relative differences but also their absolute values. In this paper, we generalize the properties of spherical distance measures between IFSs, and investigate the applications of spherical distance measures in group decision making, pattern recognition and medical diagnosis. We develop an optimization spherical distance model with IFS preference in group decision making, and demonstrate that this model is feasible and practical with an evaluation model of drought risk. By using comparative analysis method, we show that this new spherical distance can also be applied in other fields such as pattern recognition and medical diagnosis.

Keywords: spherical distance, Intuitionistic fuzzy sets, group decision-making, optimization model, drought risk analysis.

JEL Classification: C02, C44, C61, D70, D81.

Introduction

Distance measure describing the degree of difference between two sets, is fundamentally important in many natural scientific fields and social economic fields. Usually, distance measure is inversely related to similarity measure, and it can also be considered as a dual concept of similarity measure. Since the introduction of the fuzzy theory by Zadeh (1965), distance measures between fuzzy sets have gained importance due to the widespread appli-

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cations in diverse fields like approximate reasoning, risk analysis, pattern recognition, decision making, machine learning and econometric estimation, etc. (Balopoulos *et al.* 2007; Chakraborty, C., Chakraborty, D. 2006; Chen, S. M., Chen, S. H. 2009; Liu 2009, 2014; Liu, Jin 2012; Guha, Chakraborty 2010; Hyde *et al.* 2005; Ullah 1996; Vigliocco *et al.* 2002; Xu 2010, 2011; Xu, Yager 2008; Zhang, Liu 2010; Zwick *et al.* 1987). For examples, Chen, S. M. and Chen, S. H. (2009) presented a method for fuzzy risk analysis based on a similarity measure between interval-valued fuzzy numbers and new interval-valued fuzzy number arithmetic operators. Based on interval vague values, Liu (2009) proposed a technique for order preference by similarity to an ideal solution (TOPSIS) to resolve the multi-attribute decision making problem. Hyde *et al.* (2005) developed a distance-based uncertainty analysis approach to multi-criteria decision analysis for water resource decision making. Ullah (1996) provided a unified treatment of various entropy, divergence and distance measures and explored their applications in the context of econometric estimation and hypothesis testing. Balopoulos *et al.* (2007) introduced a family of normalized distance measures between binary fuzzy operators, along with its dual family of similarity measures, which were intended for applications and may be customized according to the needs and intuition of the user. Bustince and Burillo (1995, 1996) introduced the concepts of correlation and correlation coefficient of interval-valued intuitionistic fuzzy sets, they also introduced two decomposition theorems of the correlation of interval valued intuitionistic fuzzy sets.

Intuitionistic fuzzy sets (IFS) (Atanassov 1986, 1999a, 1999b) introduced by Atanassov, provides a flexible mathematical framework to cope, besides the presence of vagueness, with the hesitancy originating from imperfect or imprecise information (Pankowska, Wygralak 2006). IFS, presenting not only the membership degree, non-membership degree, but also hesitation degree, is the generalization of the classic fuzzy sets. In recent years, basic research on distance measures between IFS has also been widely studied. Atanassov (1999b) proposed the Hamming distance and Euclidean distance which are based on 2-Dimensional space. Szmidt and Kacprzyk (2000, 2005a) developed these distance measures to 3-Dimensional space. Hung and Yang (2004, 2007) presented a method to calculate the distance between IFSs on the basis of the Hausdorff distance measure, and they also proposed similarity measures of IFSs based on L_p metric. Li and Cheng (2002) introduced the new concept of the degree of similarity between IFS. Li *et al.* (2007) discussed the merits and drawbacks of different similarity measures using comparative analysis method. As discussed by Yang and Chiclana (2009), all the distances mentioned above show linear feature.

However, these distances have a common limitation that they may not adequately interpret the human perception or judgement. Let's take fuzzy sets as an example. Suppose that there is a linguistic term set $\{Perfect = 1, Good = 0.75, Fair = 0.5, Poor = 0.25, None = 0\}$ in fuzzy qualitative evaluation (Herrera *et al.* 2000). The Euclidean distance between *Perfect* and *Good* is 0.25, and the Euclidean distance between *Good* and *Fair* is also 0.25. In fact, the distance between *Perfect* and *Good* should be greater than the distance between *Good* and *Fair*. In this sense, a linear measure of distance will not be adequate to measure the difference of qualitative evaluation. In literature (Yang, Chiclana 2009), Yang and Chiclana proposed a simple nonlinear spherical distance measure between IFSs, and showed that this distance has good property. That is, the distance between any two IFSs is always lower than the distance between one IFS and the extreme crisp sets (the membership or nonmembership is 1) under the same difference of their memberships and nonmemberships. This property shows that the spherical distance model is more appropriate in measuring qualitative evaluation. In this paper, we will generalize this property.

IFS has been applied in a variety of areas such as pattern recognition (Ioannis, George 2007), logic programming (Li, Cheng 2002), medical diagnostics (De *et al.* 2001), and group decisionmaking (Pankowska, Wygralak 2006; Szmidt, Kacprzyk 2004, 2005a; Wang, Parkan 2006; Ye 2009; Zhang, Lu 2003). Group decision making is one of the active field of research in engineering and social economic analysis, it is usually performed in the presence of conflicting goals and criteria, and has to be conducted by integrating a group of experts' knowledge and experiences. When dealing with inevitably imprecise or not totally reliable judgments, IFS is especially useful. Distance measures of IFS have been used in the fields of group decision making. Most of these measures are based on linear space. For examples, Szmidt and Kacprzyk (2000, 2005b) proposed a normalized Hamming distance measure for the evaluation of a degree of agreement in a group of individuals by calculating distances between intuitionistic fuzzy preference relations. Li *et al.* (2008) proposed a fractional programming methodology on the TOPSIS (Technique for Order Preference by Similarity to an Ideal Solution) to solve Multi-attribute group decision-making problems under IFS environments. There are also lots of measures that are based on nonlinear space. For examples, Li and Ma (2008) developed a Decision Ball model to assist a decision maker in ranking alternatives and visualizing decision process. Ma (2010) also extended the Decision Ball to visualize preferences in group decisions. In this paper, we will develop a group decision making method based on IFS by constructing an optimal spherical distance model, and will utilize comparative analysis method to show the results derived by the new spherical distance are in agreement with the pattern recognition example proposed by Li and Cheng (2002), Mitchell (2003), Vlachos and Sergiadis (2007) and the medical diagnosis example by De *et al.* (2001), Szmidt and Kacprzyk (2001a, 2001b, 2002), Vlachos and Sergiadis (2007) and Wei *et al.* (2011), which demonstrate that this distance is useful and effective.

The paper is organized as follows. Section 1 gives a brief description of the IFS and its distance measure research. Section 2 discusses the properties of the spherical distance between IFSs and between their elements, and then shows the properties using examples. Section 3 applies the spherical distance model to group decision making, pattern recognition, and medical diagnosis. This section first constructs an optimization model to get the collective opinion of group decision, and then, demonstrates that the spherical distance can also be applied in many fields such as pattern recognition and medical diagnosis. A short conclusion is given in last section.

1. Problem description

An IFS (Atanassov 1986, 1999a) in X is an expression given by $A' = \{< x, u_A(x), v_A(x) > | x \in X\}$, where $u_A : X \mapsto [0,1]$, $v_A : X \mapsto [0,1]$ with the condition $0 \le u_A(x) + v_A(x) \le 1$, for all x in X. The numbers $u_A(x)$ and $v_A(x)$ denote, respectively, the membership degree and the non-membership degree of the element x in A'.

For each finite intuitionistic fuzzy set in X, $\pi_A(x)=1-u_A(x)-v_A(x)$ is called an intuitionistic fuzzy index of A'. It is a hesitation degree of whether x belongs to A' or not. It is obvious that $0\le\pi_A(x)\le1$ for each $x\in A'$. If $\pi_A(x)=0$, then $u_A(x)+v_A(x)=1$, which indicates that the intuitionistic fuzzy set A' has degenerated to the classic fuzzy set $A'=\{<x,u_A(x)>|\,x\in X\}$ (Zadeh 1965). For simplicity, $A'=\{<x,u_A(x),v_A(x)>|\,x\in X\}$ can be denoted as $A'=\{<x,u,v>\}$, where u and v are the degree of membership and the degree of non-membership, respectively, and $\pi=1-u-v$.

Atanassov (1986) introduced the Hamming distance and Euclidean distance in 2-dimension linear space, and Szmidt and Kacprzyk (2005a) generalized these distances to 3-dimension linear space. Yang and Chiclana (2009) proved that these distances have linear feature, that is, if we move both sets with the same changes in membership, nonmembership, and hesitancy degrees, then we can obtain exactly the same distance between the two IFSs. Yang and Chiclana also showed by an example that these distances in linear space can not adequately reflect the nonlinear perception of human. So they developed a simple spherical distance measure that is in nonlinear space.

Let $A'=\{<x,u,v>\}$ be an intuitionistic fuzzy set, satisfying $u+v+\pi=1$. Let $\overline{x}^2=u$, $\overline{y}^2=v$, $\overline{z}^2=\pi$, then $\overline{x}^2+\overline{y}^2+\overline{z}^2=1$. By this transformation, the intuitionistic fuzzy set A' can be viewed as a spherical surface $S=\{(\overline{x},\overline{y},\overline{z})\,|\,\overline{x}^2+\overline{y}^2+\overline{z}^2=1\}$, and the distance between two elements of the intuitionistic fuzzy set can be defined as the spherical distance between their corresponding points on its spherical surface representation.

Let $x_A=<x_A,u_A,v_A>$ and $x_B=<x_B,u_B,v_B>$ be two elements of A', and let $A(\overline{x}_A,\overline{y}_A,\overline{z}_A)$, $B(\overline{x}_B,\overline{y}_B,\overline{z}_B)$ satisfying $\overline{x}_A^2+\overline{y}_A^2+\overline{z}_A^2=1$, $\overline{x}_B^2+\overline{y}_B^2+\overline{z}_B^2=1$ be two points on the spherical surface S. It follows from the transformation above that $u_A=\overline{x}_A^2$, $v_A=\overline{y}_A^2$, $\pi_A=\overline{z}_A^2$, $u_B=\overline{x}_B^2$, $v_B=\overline{y}_B^2$ and $\pi_B=\overline{z}_B^2$. The shortest path between points A and B are defined as the length of the arc of the great circle passing through both points, which is called the spherical distance between points A and B:

$$D(A,B)=arccos\{1-1/2[(\overline{x}_A-\overline{x}_B)^2+(\overline{y}_A-\overline{y}_B)^2+(\overline{z}_A-\overline{z}_B)^2]\}. \tag{1}$$

For

$$D(A,B)=arccos((u_Au_B)^{\frac{1}{2}}+(v_Av_B)^{\frac{1}{2}}+(\pi_A\pi_B)^{\frac{1}{2}}). \tag{2}$$

This distance can be interpreted as the spherical distance between two elements of the IFS.

The spherical distance between two IFSs, $A'=\{<x_i,u_A(x_i),v_A(x_i)>|\,x_i\in X\}$ and $B'=\{<x_i,u_B(x_i),v_B(x_i)>|\,x_i\in X\}$ of the universe of discourse $X=\{x_1,x_2,\ldots,x_n\}$ is also defined as follows:

$$D(A',B')=\frac{2}{n\pi}\sum_{i=1}^{n}arccos\{1-\frac{1}{2}[(\sqrt{u_A(x_i)}-\sqrt{u_B(x_i)})^2+ \tag{3}$$

$$(\sqrt{v_A(x_i)}-\sqrt{v_B(x_i)})^2+(\sqrt{\pi_A(x_i)}-\sqrt{\pi_B(x_i)})^2]\},$$

where $\frac{2}{\pi}$ is introduced to get distance values in the range $[0,1]$ instead of $[0,\frac{\pi}{2}]$. Eq. (3) is equivalent to

$$D(A',B') = \frac{2}{n\pi}\sum_{i=1}^{n} arccos((u_A(x_i)u_B(x_i))^{\frac{1}{2}} + (v_A(x_i)v_B(x_i))^{\frac{1}{2}} + (\pi_A(x_i)\pi_B(x_i))^{\frac{1}{2}}). \quad (4)$$

The spherical distance measure between IFSs has good property, which has been shown by Yang and Chiclana (2009).

Property 1 (Yang, Chiclana 2009). Let $A' = \{< x_i, u_A(x_i), v_A(x_i) > | x_i \in X\}$ and $B' = \{< x_i, u_B(x_i), v_B(x_i) > | x_i \in X\}$ and $E' = \{< x_i, u_E(x_i) = 1, v_E(x_i) = 0 > | x_i \in X\}$ or $< x_i, u_E(x_i) = 0, v_E(x_i) = 1 > | x_i \in X\}$ be three intuitionistic fuzzy sets, and let $a = \{a_1, a_2, \ldots, a_n\}$, $b = \{b_1, b_2, \ldots, b_n\}$ be two sets of real positive numbers satisfying

$$| u_B(x_i) - u_A(x_i) | = a_i, | v_B(x_i) - v_A(x_i) | = b_i,$$
$$| u_E(x_i) - u_A(x_i) | = a_i, | v_E(x_i) - v_A(x_i) | = b_i$$

then the inequality $D(A',B') < D(A',E')$ holds.

Property 1 can be explained that "the distance between two IFSs is always lower than the distance between any one of IFSs and the extreme crisp sets (E') under the same difference of their memberships and nonmemberships". If we consider the distance between any two elements of IFS, then Property 1 actually implies Property 2 as follows.

Property 2. Let $x_A =< x_A, u_A, v_A >$, $x_B =< x_B, u_B, v_B >$ and $x_E =< x_E, u_E = 1, v_E = 0 >$ or $x_E =< x_E, u_E = 0, v_E = 1 >$ be three elements of A'. If $| u_B - u_A | = | a |$, $| v_B - v_A | = | b |$, and $| u_E - u_A | = | a |$, $| v_E - v_A | = | b |$, then the inequality $D(A,B) < D(A,E)$ holds.

Property 2 can be explained that "the distance between two elements of IFS is always lower than the distance between any one element of IFS and the extreme element (x_E) under the same difference of their memberships and nonmemberships".

Let us consider the linguistic term of intuitionistic fuzzy set $\{Perfect =< s_1, 1, 0 >;$ $Good =< s_2, 0.75, 0.25 >; Fair =< s_3, 0.5, 0.5 >; Poor =< s_4, 0.25, 0.75 >; None =< s_5, 0, 1 >\}$, which is equivalent to the linguistic term of fuzzy set $\{Perfect = 1, Good = 0.75, Fair = 0.5,$ $Poor = 0.25, None = 0\}$. The Euclidean distances of the membership degree, nonmembership degree, and the spherical distances between the linguistic term of intuitionistic fuzzy set are shown in Table 1.

Table 1. The spherical distances between the linguistic term of IFSs

The Euclidean distances of the membership degree	The Euclidean distances of the nonmembership degree	The spherical distances	The relation between the spherical distances
$\lvert \mu'_{s_1} - \mu'_{s_2} \rvert = 0.25$	$\lvert \nu'_{s_1} - \nu'_{s_2} \rvert = 0.25$	$D(s_1, s_2) = 0.5236$	$D(s_1, s_2) > D(s_2, s_3)$
$\lvert \mu'_{s_2} - \mu'_{s_3} \rvert = 0.25$	$\lvert \nu'_{s_2} - \nu'_{s_3} \rvert = 0.25$	$D(s_2, s_3) = 0.2618$	
$\lvert \mu'_{s_3} - \mu'_{s_4} \rvert = 0.25$	$\lvert \nu'_{s_3} - \nu'_{s_4} \rvert = 0.25$	$D(s_3, s_4) = 0.2618$	$D(s_4, s_5) > D(s_3, s_4)$
$\lvert \mu'_{s_4} - \mu'_{s_5} \rvert = 0.25$	$\lvert \nu'_{s_4} - \nu'_{s_5} \rvert = 0.25$	$D(s_4, s_5) = 0.5236$	

According to Table 1, the Euclidean distance between *Perfect* and *Good* is equivalent to the Euclidean distance between *Good* and *Fair*, while the spherical distance between *Perfect* and *Good* is greater than the spherical distance between *Good* and *Fair*. For instance, it is easy for a student to improve his test score from a *Fair* level (For example, the sore is 60) to a *Good* level (80). But he or she must put more effort to reach a *Perfect* level (100) from a *Good* level (80). In this sense, the spherical distance measure is more appropriate in measuring qualitative evaluation than the usual Euclidean distance measure.

In fact, we can enlarge Property 2 to more general cases. For examples, let us consider Table 2 and Table 3 below, where $x_{A_i} =< x_{A_i}, u'_{A_i}, v'_{A_i} >$ and $x_{B_i} =< x_B, \mu'_{B_i}, \nu'_{B_i} >$ are two elements of A'. When the Euclidean distance satisfying $u'_{B_i} - u'_{A_i} = -0.1$, $\nu'_{B_i} - \nu'_{A_i} = 0.2$, the inequality $D(A_i, B_i) > D(A_j, B_j)$ (or $D(A_i, B_i) < D(A_j, B_j)$) hold for all $i, j = \{1,2,3,4\}, i < j$.

Table 2. An numerical example of the spherical distances based on equal Euclidean distances

	μ'_{A_i}	ν'_{A_i}	μ'_{B_i}	ν'_{B_i}	a	b	$D(A_i, B_i)$
$i = 1$	0.1	0.2	0.0	0.4	–0.1	0.2	0.3739
$i = 2$	0.2	0.2	0.1	0.4	–0.1	0.2	0.2373
$i = 3$	0.3	0.2	0.2	0.4	–0.1	0.2	0.2241
$i = 4$	0.4	0.2	0.3	0.4	–0.1	0.2	0.2211

Table 3. Another numerical example of the spherical distances based on equal Euclidean distances

	μ'_{A_i}	ν'_{A_i}	μ'_{B_i}	ν'_{B_i}	a	b	$D(A_i, B_i)$
$i = 1$	0.5	0.2	0.4	0.4	–0.1	0.2	0.2241
$i = 2$	0.6	0.2	0.5	0.4	–0.1	0.2	0.2373
$i = 3$	0.7	0.2	0.6	0.4	–0.1	0.2	0.3739

In next section, we will show that for any two elements $x_{A_i} =< x_{A_i}, u_{A_i}, v_{A_i} >$ and $x_{B_i} =< x_B, u_{B_i}, v_{B_i} >$ of A', if $|u_{B_i} - u_{A_i}| = |a|$, $|v_{B_i} - v_{A_i}| = |b|$, then the inequality $D(A_i, B_i) > D(A_j, B_j)$ (or $D(A_i, B_i) < D(A_j, B_j)$) hold, where $i, j \in M = \{1, 2, \ldots, m\}, i < j$. Because this property is suitable for measuring qualitative evaluation, we will apply the spherical distance measure to group decision making in Section 3.

2. The properties of the Spherical distance between any two elements of the IFS

Let $x_A =< x_A, u_A, v_A >$ and $x_B =< x_B, u_B, v_B >$ be two elements of A', and let A, B be two corresponding points on the spherical surface S. If we denote $u_B = u_A + a$, $v_B = v_A + b$, and $u_A = u, v_A = v$, where $a, b \in R$ satisfy $u + v + a + b \le 1$. Then $D(A, B)$ can be denoted as

$$D(A, B) = d(u, v) = \arccos f(u, v), \tag{5}$$

where

$$f(u, v) = (u(u+a))^{\frac{1}{2}} + (v(v+b))^{\frac{1}{2}} + ((1-u-v)(1-u-v-a-b))^{\frac{1}{2}} \ . \tag{6}$$

In the following, we will discuss the properties of $d(u,v)$. We first introduce Lemma 1.

Lemma 1 (Hamdy 2007)**.** Let $C \subset R^n$ be a nonempty open convex set, and let $f: C \subset R^n \mapsto R$ be twice continuously differentiable over R^n. f is concave if and only if its Hessian matrix $\nabla^2 f(\mathbf{x})$ is negative semi-definite at each point in C. f is strictly concave if its Hessian matrix $\nabla^2 f(\mathbf{x})$ is negative definite at each point in C.

Lemma 2. $f(u,v)=(u(u+a))^{\frac{1}{2}}+(v(v+b))^{\frac{1}{2}}+((1-u-v)(1-u-v-a-b))^{\frac{1}{2}}$ is concave over $C=\{(u,v)\in R^2 \mid 0\le u\le 1, 0\le v\le 1, 0\le u+v\le 1, 0\le u+v+a+b\le 1\}$, where $0\le a,b\le 1$, and a,b are not both equal to zero.

Proof. Let's consider $f'_u(u,v)$, $f'_v(u,v)$, $f''_{uu}(u,v)$, $f''_{vv}(u,v)$ and $f''_{uv}(u,v)$, where

$$f'_u(u,v)=\frac{1}{2}(2u+a)(u(u+a))^{-\frac{1}{2}}+\frac{1}{2}(2u-2+2v+a+b)((1-u-v)(1-u-v-a-b))^{-\frac{1}{2}}; \quad (7)$$

$$f'_v(u,v)=\frac{1}{2}(2v+b)(v(v+b))^{-\frac{1}{2}}+\frac{1}{2}(2u-2+2v+a+b)((1-u-v)(1-u-v-a-b))^{-\frac{1}{2}}; \quad (8)$$

$$f''_{uu}(u,v)=-a^2(u(u+a))^{-3/2}/4-(a+b)^2((1-u-v)(1-u-v-a-b))^{-3/2}/4; \quad (9)$$

$$f''_{vv}(u,v)=-b^2(v(v+b))^{-3/2}/4-(a+b)^2((1-u-v)(1-u-v-a-b))^{-3/2}/4 \quad (10)$$

and

$$f''_{uv}(u,v)=-(a+b)^2((1-u-v)(1-u-v-a-b))^{-3/2}/4. \quad (11)$$

Obviously, we have $f''_{uu}(u,v)\le 0, f''_{vv}(u,v)\le 0$ and $f''_{uv}(u,v)\le 0$. $f''_{uu}(u,v), f''_{vv}(u,v)$ and $f''_{uv}(u,v)$ are continuous. If we denote $y_1=(u(u+a))^{-3/2}/4$, $y_2=(v(v+b))^{-3/2}/4$, and $y=((1-u-v)(1-u-v-a-b))^{-3/2}/4$, then the Hessian matrix of $f(u,v)$ is $\nabla^2 f(u,v)=\begin{pmatrix} f_{uu}(u,v) & f_{uv}(u,v) \\ f_{uv}(u,v) & f_{vv}(u,v)\end{pmatrix}$, and the determinant of $\nabla^2 f(u,v)$ satisfies $det(\nabla^2 f(u,v))=a^2b^2y_1y_2+(a+b)^2(a^2y_1+b^2y_2)y>0$. That is, $\nabla^2 f(u,v)$ is negative definite for all $(u,v)\in C$. By lemma 1, $f(u,v)$ is concave.

According to Lemma 2, for any given u_0, $0\le u_0\le 1$, $f(v)=(u_0(u_0+a))^{\frac{1}{2}}+(v(v+b))^{\frac{1}{2}}+((1-u_0-v)(1-u_0-v-a-b))^{\frac{1}{2}}$ is concave. For any given v_0, $0\le v_0\le 1$, $f(u)=(u(u+a))^{\frac{1}{2}}+(v_0(v_0+b))^{\frac{1}{2}}+((1-u-v_0)(1-u-v_0-a-b))^{\frac{1}{2}}$ is concave.

For any given v_0, $0\le v_0\le 1$, let
$d(u)=arccos(f(u))=arccos((u(u+a))^{\frac{1}{2}}+(v_0(v_0+b))^{\frac{1}{2}}+((1-u-v_0)(1-u-v_0-a-b))^{\frac{1}{2}})$,
where $u\in\{u \mid 0\le u\le 1, 0\le u+v_0\le 1, 0\le u+v_0+a+b\le 1\}$, $0\le a,b\le 1$,
and a,b are not both equal to zero. For $f''(u)\le 0$, we can easily get that
$|f''(u)(1-f^2(u))|\ge|f'(u)^2f(u)|$, then the second derivative $d''(u)=(-f''(u)(1-f^2(u))-f'(u)^2f(u))((1-f^2(u))^{-\frac{3}{2}})\ge 0$, which denotes that $d(u)$ is convex. For any given u_0, let
$d(v)=arccos(f(v))=arccos((u_0(u_0+a))^{\frac{1}{2}}+(v(v+b))^{\frac{1}{2}}+((1-u_0-v)(1-u_0-v-a-b))^{\frac{1}{2}})$,
where $v\in\{v \mid 0\le v\le 1, 0\le u_0+v\le 1, 0\le u_0+v+a+b\le 1\}$, $0\le a,b\le 1$, and a,b are not both equal to zero. For $f''(u)\le 0$, we can easily get that $|f''(u)(1-f^2(u))|\ge|f'(u)^2f(u)|$,

then the second derivative $d''(u)=(-f''(u)(1-f^2(u))-f'(u)^2 f(u))((1-f^2(u))^{-\frac{3}{2}})\geq 0$, which denotes that $d(u)$ is convex. For any given u_0, let
$d(v)=arccos(f(v))=arccos((u_0(u_0+a))^{\frac{1}{2}}+(v(v+b))^{\frac{1}{2}}+((1-u_0-v)(1-u_0-v-a-b))^{\frac{1}{2}})$,
where $v\in\{v\,|\,0\leq v\leq 1, 0\leq u_0+v\leq 1, 0\leq u_0+v+a+b\leq 1\}$, $0\leq a,b\leq 1$, and a,b are not both equal to zero. For $f''(v)\leq 0$, we can easily get that $|f''(v)(1-f^2(v))|\geq|f'(v)^2 f(v)|$, then the second derivative $d''(v)=(-f''(v)(1-f^2(v))-f'(v)^2 f(v))((1-f^2(v))^{-\frac{3}{2}})\geq 0$, which denotes that $d(v)$ is convex.

Let $f'_u(u,v)=0$, and $f'_v(u,v)=0$, then we have

$$u=a(v-1)/b, u=a(1-a-b-v)/(2a+b); \tag{12}$$

$$v=a(u-1)/b, v=a(1-a-b-u)/(2b+a). \tag{13}$$

Let $x_i=<x_i,u_i,v_i>$ and $x_{i'}=<x_{i'},u_{i'},v_{i'}>$ be the elements of the intuitionistic fuzzy sets, and their corresponding points on the spherical surface *S* are $A_{i'}$ and A_i, respectively, where $u_{i'}=u_i+a$, $v_{i'}=v_i+b$, $i,i'\in N$. The spherical distance between A_i and $A_{i'}$ are

$$D(A_i,A_{i'})=d(u_i,v_i)=\arccos f(u_i,v_i)=$$
$$\arccos((u_i(u_i+a))^{\frac{1}{2}}+(v_i(v_i+b))^{\frac{1}{2}}+((1-u_i-v_i)(1-u_i-v_i-a-b))^{\frac{1}{2}}), i,i'\in N$$

and the Euclidean distance between $A_{i'}$ and A_i, $i\in N$, are $(a^2+b^2+(a+b)^2)^{\frac{1}{2}}$.

The other two properties of function $d(u,v)$ are as follows.

Property 3. The convex function $d(u,v_0)$ (or $d(u_0,v)$) is increasing on the interval $[\xi,\xi_2]$, $d(u,v_0)$ (or $d(u_0,v)$) is decreasing on the interval $[\xi_1,\xi]$, where $[\xi_1,\xi_2]$ is the domain ($[\xi_1,\xi_2]=\{u\,|\,0\leq u\leq 1, 0 lequ+v_0\leq 1, 0\leq u+v_0+a+b\leq 1\}$) of $d(u,v_0)$ (or $d(u_0,v)$), ξ is the minimum point (That is, $d(\xi,v_0)\leq d(u,v_0), u\in[\xi_1,\xi_2]$.), and v_0 (or u_0) is a constant. This denotes that, for any $i>j$, $i,j,i',j'\in N$, if $u_i>u_j\geq\xi$, then $D(A_i,A_{i'})>D(A_j,A_{j'})$; for any $i<j$, $i,j,i',j'\in N$, if $u_i<u_j\leq\xi$, then $D(A_i,A_{i'})>D(A_j,A_{j'})$.

Property 3 denotes that: For a fixed Euclidean distance $(a^2+b^2+(a+b)^2)^{\frac{1}{2}}$ between $A_{i'}$ and A_i, the smaller Euclidean distance between u_i and left-endpoint ξ_1 (or right-endpoint ξ_2) of the interval $[\xi_1,\xi_2]$ is, the greater spherical distance between $A_{i'}$ and A_i, $i,i'\in N$ is.

Property 4. $d'_u(u,v_0)$ (or $d'_v(u_0,v)$) is increasing on the interval $[\xi,\xi_2]$; $d'_u(u,v_0)$ (or $d'_v(u_0,v)$) is decreasing on the interval $[\xi_1,\xi]$, where ξ is the minimum point (That is, $d(\xi,v_0)\leq d(u,v_0), u\in[\xi_1,i_2]=\{u\,|\,0\leq u\leq 1, 0\leq u+v_0\leq 1, 0\leq u+v_0+a+b\leq 1\}$.). This denotes that, for any $i<j<l<m$, $i,j,l,m,i',j',l',m'\in N$, if $u_i<u_j<u_l<u_m\leq\xi$, and $u_j-u_i=u_m-u_l=k_0$, then $D(A_l,A_{l'})-D(A_m,A_{m'})<D(A_i,A_{i'})-D(A_j,A_{j'})$; for any $i<j<l<m$, $i,j,l,m,i',j',l',m'\in N$, if $\xi<u_i<u_j<u_l<u_m$, and $u_j-u_i=u_m-u_l=k_0$, then $D(A_i,A_{i'})-D(A_j,A_{j'})>D(A_l,A_{l'})-D(A_m,A_{m'})$.

Property 4 denotes that: For a fixed Euclidean distance $(a^2+b^2+(a+b)^2)^{\frac{1}{2}}$ between $A_{i'}$ and A_i, the smaller Euclidean distance between u_i and left-endpoint (or right-endpoint) of the interval $[\xi_1,\xi_2]$ is, the greater spherical distance difference between $D(A_i,A_{i'})$ and

$D(A_j, A_{j'})$, where $u_j - u_i = k_0$, $i, j, i', j' \in N$. Property 3 and Property 4 show that spherical distance difference is nonlinear, which can better reflect the character of human perception and judgement. Property 3 and Property 4 are actually the generalization of Property 2.

The two properties can be explained by four different cases. For simplicity, we consider six points $A_i, i = 1, \ldots, 6$, and we also denote $u_{j+3} = u_j + a$, $v_j = v_0$, $v_{j+3} = v_j + b$, $j = 1,2,3$. The Euclidean distances between A_{j+3} and A_j, $j = 1,2,3$ are $(a^2 + b^2 + (a+b)^2)^{\frac{1}{2}}$.

Case 1. $a \geq 0$, $b \geq 0$, and a, b are not both equal to zero (That is, if $a = 0$, then $b > 0$; if $b = 0$, then $a > 0$.). For a given number $v_0 \in [0, 1-a-b]$ (We can also get the similar conclusion with $u_0 \in [0, 1-a-b]$).

- $d(u, v_0)$ is monotone increasing on the interval $[a(1-a-b-v_0)/(2a+b), 1-a-b-v_0]$, that is, when $1-a-b-v_0 \geq u_3 > u_2 > u_1 \geq a(1-a-b-v_0)/(2a+b)$, we have $D(A_1, A_4) < D(A_2, A_5) < D(A_3, A_6)$; $d'(u, v_0)$ is monotone increasing on the interval $[a(1-a-b-v_0)/(2a+b), 1-a-b-v_0]$, that is, when $1-a-b-v_0 \geq u_3 > u_2 > u_1 \geq a(1-a-b-v_0)/(2a+b)$, and $u_3 - u_2 = u_2 - u_1$, we have $D(A_2, A_5) - D(A_1, A_4) < D(A_3, A_6) - D(A_2, A_5)$.
- $d(u, v_0)$ is monotone decreasing on the interval $[0, a(1-a-b-v_0)/(2a+b)]$, that is, when $a(1-a-b-v_0)/(2a+b) \geq u_3 > u_2 > u_1 > 0$, we have $D(A_1, A_4) > D(A_2, A_5) > D(A_3, A_6)$; $d'(u, v_0)$ is monotone increasing on the interval $[0, a(1-a-b-v_0)/(2a+b)]$, that is, when $a(1-a-b-v_0)/(2a+b) \geq u_3 > u_2 > u_1 > 0$, and $u_3 - u_2 = u_2 - u_1$, we have $D(A_1, A_4) - D(A_2, A_5) > D(A_2, A_5) - D(A_3, A_6)$.

Proof. By Eqs (12) and (13), it is clear that $u = a(v-1)/b < 0$, and $v = a(u-1)/b < 0$. Therefore, $u = a(1-a-b-v)/(2a+b)$ and $v = a(1-a-b-u)/(2b+a)$ are the solutions to $f'_u(u,v) = 0$ and $f'_v(u,v) = 0$, respectively.

Let v_0, $v_0 \in [0, 1-a-b]$ be a given number. We have $f(a(1-a-b-v_0)/(2a+b), v_0) = 0$, and $f''_{uu}(a(1-a-b-v_0)/(2a+b), v_0) \leq 0$. It follows that $(a(1-a-b-v_0)/(2a+b), v_0)$ is the maximum point of the concave function $f(u, v_0)$, which also denotes that $(a(1-a-b-v_0)/(2a+b), v_0)$ is the minimum point of the convex function $d(u, v_0)$. Therefore, $d(u, v_0)$ is monotone increasing on the interval $[a(1-a-b-v_0)/(2a+b), 1-a-b-v_0]$, and monotone decreasing on the interval $[0, a(1-a-b-v_0)/(2a+b)]$. If $1-a-b-v_0 \geq u_3 > u_2 > u_1 \geq a(1-a-b-v_0)/(2a+b)$, then $d(u_3, v_0) > d(u_2, v_0) > d(u_1, v_0)$, that is, $D(A_3, A_6) > D(A_2, A_5) > D(A_1, A_4)$; If $a(1-a-b-v_0)/(2a+b) \geq u_3 > u_2 > u_1 > 0$, then $d(u_1, v_0) > d(u_2, v_0) > d(u_3, v_0)$, that is, $D(A_1, A_4) > D(A_2, A_5) > D(A_3, A_6)$.

Suppose that $u_3 - u_2 = u_2 - u_1$. For $d(u, v_0)$ is convex, we get $d''_{uu}(u, v_0) > 0$. It follows that $d'_u(u, v_0)$ is monotone increasing on the interval $[a(1-a-b-v_0)/(2a+b), 1-a-b-v_0]$. Thus we have $\dfrac{d(u_2, v_0) - d(u_1, v_0)}{u_2 - u_1} < \dfrac{d(u_3, v_0) - d(u_2, v_0)}{u_3 - u_2}$.

For $d(u_1, v_0) < d(u_2, v_0) < d(u_3, v_0)$, we have $d(u_2, v_0) - d(u_1, v_0) > 0$, and $d(u_3, v_0) - d(u_2, v_0) > 0$. Therefore, $d(u_2, v_0) - d(u_1, v_0) < d(u_3, v_0) - d(u_2, v_0)$.

That is, $D(A_2, A_5) - D(A_1, A_4) < D(A_3, A_6) - D(A_2, A_5)$.

When $0 \leq u < a(1-a-b-v_0)/(2a+b))$, we also have $d'_u(u, v_0)$ is a monotone increasing function. Thus we have $\dfrac{d(u_2, v_0) - d(u_1, v_0)}{u_2 - u_1} < \dfrac{d(u_3, v_0) - d(u_2, v_0)}{u_3 - u_2}$. For

$d(u_1,v_0) > d(u_2,v_0) > d(u_3,v_0)$, we have $d(u_2,v_0) - d(u_1,v_0) < 0$, and $d(u_3,v_0) - d(u_2,v_0) < 0$. Therefore, $d(u_1,v_0) - d(u_2,v_0) > d(u_2,v_0) - d(u_3,v_0)$. That is, $D(A_1,A_4) - D(A_2,A_5) > D(A_2,A_5) - D(A_3,A_6)$.

Example 1. Let's consider two elements $x_A =< x_A, u_A, v_A >$, $x_B =< x_B, u_B, v_B >$ of the IFS A', where $u_B = u_A + 0.1$, $v_B = v_A + 0.05$, $v_A = 0.2$. We denote $u_A = u$ and $v_A = v$. The spherical distance between points A and B are

$$d(u,0.2) = \arccos((u(u+0.1)^{\frac{1}{2}} + (0.2(0.2+0.05)^{\frac{1}{2}} + ((1-u-0.2)(1-u-0.2-0.1-0.05)^{\frac{1}{2}}), \tag{14}$$

$d(u,0.2)$ is actually an intersecting line between the 3-D surface $d(u,v) = \arccos((u(u+0.1))^{\frac{1}{2}} + (v(v+0.05))^{\frac{1}{2}} + ((1-u-v)(1-u-v-0.1-0.05))^{\frac{1}{2}})$ and 2-D plane $v = 0.2$. (As shown in Fig. 1 and 2). $d(u,0.2)$ is a convex function, and its lowest point is $(0.26,0.2)$. $d(u,0.2)$ is decreasing on the interval $[0,0.26]$, and it is increasing on the interval $[0.26,0.65]$ (As shown in Fig. 2).

For any $0.65 \geq u_3 > u_2 > u_1 \geq 0.26$, and $u_3 - u_2 = u_2 - u_1$, we have $d(u_3,v_0) > d(u_2,v_0) > d(u_1,v_0)$, and $d(u_2,v_0) - d(u_1,v_0) < d(u_3,v_0) - d(u_2,v_0)$. For any $0.26 \geq u_3 > u_2 > u_1 > 0$, and $u_3 - u_2 = u_2 - u_1$, we have $d(u_1,v_0) > d(u_2,v_0) > d(u_3,v_0)$, and $d(u_1,v_0) - d(u_2,v_0) > d(u_2,v_0) - d(u_3,v_0)$. This denotes that $d(u,0.2)$ is decreasing quickly near the left endpoint (0) of the interval $[0,0.65]$, and $d(u,0.2)$ is increasing quickly near the right endpoint (0.65) of the interval $[0,0.65]$ (As shown in Fig. 2).

Case 2. $a \leq 0$, $b \leq 0$, and a, b are not both equal to zero. Let v_0, $v_0 \in [-b, 1+a]$ be a given number (We can also get the similar conclusion with u_0, $u_0 \in [-a, 1+b]$). We only need to consider two intervals $u \in [a(1-a-b-v_0)/(2a+b), 1-v_0]$ and $u \in [-a, a(1-a-b-v_0)/(2a+b)]$, respectively. The conclusion is similar to Case 1.

Case 3. $a \leq 0$, $b \geq 0$, and a, b are not both equal to zero.

Subcase 3.1. $|a| \geq b$. Let $v_0 \in [0, 1+a]$ be a given number (We can also get the similar conclusion with $u_0 \in [-a, 1]$).

- $d(u,v_0)$ is monotone increasing on interval $[a(1-a-b-v_0)/(2a+b), 1-v_0]$, that is, when $1 - v_0 \geq u_3 > u_2 > u_1 \geq a(1-a-b-v_0)/(2a+b)$, we have $D(A_1,A_4) < D(A_2,A_5) < D(A_3,A_6)$; $d'(u,v_0)$ is monotone increasing on the interval $[a(1-a-b-v_0)/(2a+b), 1-v_0]$, that is, when $1 - v_0 \geq u_3 > u_2 > u_1 \geq a(1-a-b-v_0)/(2a+b)$, and $u_3 - u_2 = u_2 - u_1$, we have $D(A_2,A_5) - D(A_1,A_4) < D(A_3,A_6) - D(A_2,A_5)$.
- $d(u,v_0)$ is monotone decreasing on the interval $[-a, a(1-a-b-v_0)/(2a+b)]$, that is, when $a(1-a-b-v_0)/(2a+b) \geq u_3 > u_2 > u_1 \geq -a$, we have $D(A_1,A_4) > D(A_2,A_5) \geq D(A_3,A_6)$; $d'(u,v_0)$ is monotone increasing on the interval $[-a, a(1-a-b-v_0)/(2a+b)]$, that is, when $a(1-a-b-v_0)/(2a+b) \geq u_3 > u_2 > u_1 \geq -a$, and $u_3 - u_2 = u_2 - u_1$, we have $D(A_1,A_4) - D(A_2,A_5) > D(A_3,A_6)$.

Subcase 3.2. $|a| < b$. For a given number $v_0 \in [0, 1-b]$. (We can also get the similar conclusion with $u_0 \in [-a, 1-a-b]$).

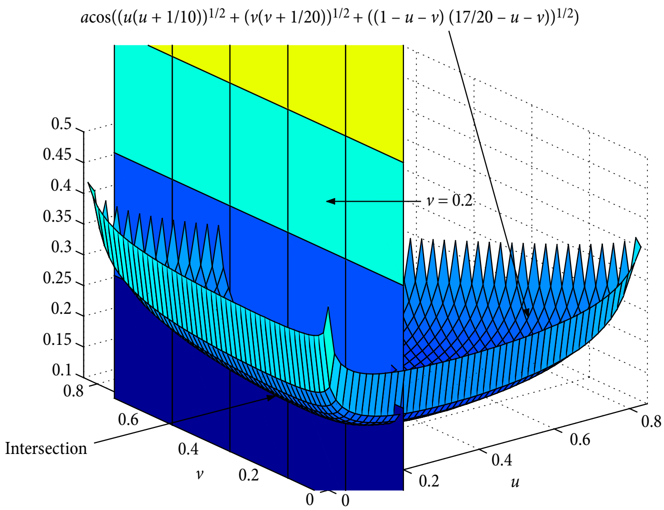

Fig. 1. The intersection between $d(u,v)$ and $v = 0.2$

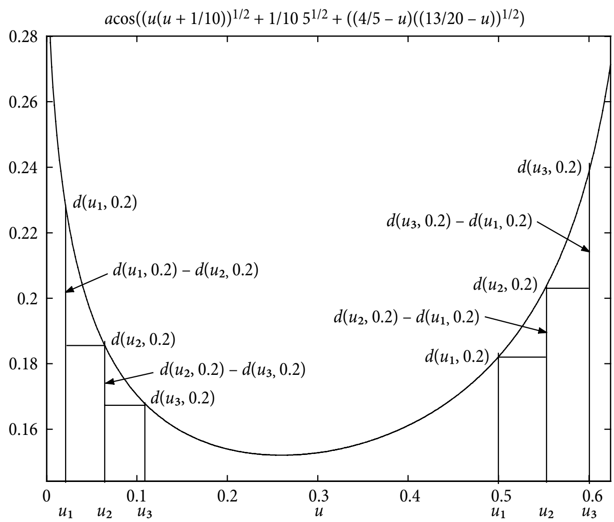

Fig. 2. Figure of function $d(u,0.2)$

- $d(u,v_0)$ is monotone increasing on the interval $[a(v_0-1)/b, 1-v_0-a-b]$, that is, when $1-v_0-a-b \geq u_3 > u_2 > u_1 \geq a(v_0-1)/b$, we have $D(A_1,A_4) < D(A_2,A_5) < D(A_3,A_6)$; $d'(u,v_0)$ is monotone increasing on the interval $[a(v_0-1)/b, 1-v_0-a-b]$, that is, when $1-v_0-a-b \geq u_3 > u_2 > u_1 \geq a(v_0-1)/b$, and $u_3-u_2 = u_2-u_1$, we have $D(A_2,A_5)-D(A_1,A_4) < D(A_3,A_6)-D(A_2,A_5)$.
- $d(u,v_0)$ is monotone decreasing on the interval $[-a, a(v_0-1)/b]$, that is, when $a(v_0-1)/b \geq u_3 > u_2 > u_1 \geq -a$, we have $D(A_1,A_4) > D(A_2,A_5) > D(A_3,A_6)$; $d'(u,v_0)$ is monotone increasing on the interval $[-a, a(v_0-1)/b]$, that is, when $a(v_0-1)/b \geq u_3 > u_2 > u_1 \geq -a$, and $u_3-u_2 = u_2-u_1$, we have $D(A_1,A_4)-D(A_2,A_5) > D(A_2,A_5)-D(A_3,A_6)$.

Proof. Subcase 3.1. $|a| \geq b$. For $u - a(v-1)/b < u + v - 1 \leq 0$, we have $u = a(1-a-b-v)/(2a+b)$.

Let v_0, $v_0 \in [0, 1+a]$ be a given number. We have $f(a(1-a-b-v_0)/(2a+b), v_0) = 0$, and $f''_{uu}(a(1-a-b-v_0)/(2a+b), v_0) \leq 0$. It follows that $(a(1-a-b-v_0)/(2a+b), v_0)$ is the maximum point of the concave function $f(u, v_0)$, which also denotes that $(a(1-a-b-v_0)/(2a+b), v_0)$ is the minimum point of the convex function $d(u, v_0)$. Therefore, $d(u, v_0)$ is monotone increasing on the interval $[a(1-a-b-v_0)/(2a+b), 1-v_0]$, and $d(u, v_0)$ is monotone decreasing on the interval $[-a, a(1-a-b-v_0)/(2a+b)]$. If $1 - v_0 \geq u_3 > u_2 > u_1 \geq a(1-a-b-v_0)/(2a+b)$, then $d(u_3, v_0) > d(u_2, v_0) > d(u_1, v_0)$. Thus we have $D(A_3, A_6) > D(A_2, A_5) > D(A_1, A_4)$. If $a(1-a-b-v_0)/(2a+b) \geq u_3 > u_2 > u_1 \geq -a$, then $d(u_1, v_0) > d(u_2, v_0) > d(u_3, v_0)$. Thus we have $D(A_1, A_4) > D(A_2, A_5) > D(A_3, A_6)$.

Suppose that $u_3 - u_2 = u_2 - u_1$. For $d(u, v_0)$ is concave, we get $d''_{uu}(u, v_0) > 0$. It follows that $d'_u(u, v_0)$ is monotone increasing on the interval $[a(1-a-b-v_0)/(2a+b), 1-v_0]$, thus we have $\frac{d(u_2, v_0) - d(u_1, v_0)}{u_2 - u_1} < \frac{d(u_3, v_0) - d(u_2, v_0)}{u_3 - u_2}$. Then for $d(u_1, v_0) < d(u_2, v_0) < d(u_3, v_0)$, we have $d(u_2, v_0) - d(u_1, v_0) > 0$, and $d(u_3, v_0) - d(u_2, v_0) > 0$. Therefore, $d(u_2, v_0) - d(u_1, v_0) < d(u_3, v_0) - d(u_2, v_0)$. That is, $D(A_2, A_5) - D(A_1, A_4) < D(A_3, A_6) - D(A_2, A_5)$.

When $-a \leq u < a(1-a-b-v_0)/(2a+b)$, we also have $d'_u(u, v_0)$ is a monotone decreasing function. It follows that $\frac{d(u_2, v_0) - d(u_1, v_0)}{u_2 - u_1} < \frac{d(u_3, v_0) - d(u_2, v_0)}{u_3 - u_2}$, then for $d(u_1, v_0) > d(u_2, v_0) > d(u_3, v_0)$, we have $d(u_2, v_0) - d(u_1, v_0) < 0$, and $d(u_3, v_0) - d(u_2, v_0) < 0$. Therefore, $d(u_1, v_0) - d(u_2, v_0) > d(u_2, v_0) - d(u_3, v_0)$. That is, $D(A_1, A_4) - D(A_2, A_5) > D(A_2, A_5) - D(A_3, A_6)$.

Subcase 3.3. $|a| < b$. When $|a| < b/2$. For $u - a(1-a-b-v)/(2a+b) > 0$, we have $u = a(v-1)/b$. When $b/2 \leq |a| < b$. For $u \geq -a$, and $u + v < 1$, we have $v < 1 - u < 1 + a$ and $a(1-a-b-v)/(2a+b) > 1 - v$. Thus we have $u - a(1-a-b-v)/(2a+b) < u + v - 1 < 0$. Therefore, when $|a| < b$, we have $u = a(v-1)/b$.

Let v_0, $v_0 \in [0, 1-b]$ be a given number. We have $f(a(v_0 - 1)/b, v_0) = 0$, and $f''_{uu}(a(v_0 - 1)/b, v_0) \leq 0$. It follows that $(a(v_0 - 1)/b, v_0)$ is the maximum point of the concave function $f(u, v_0)$, which also denotes that $(a(v_0 - 1)/b, v_0)$ is the minimum point of the convex function $d(u, v_0)$. Therefore, $d(u, v_0)$ is monotone increasing on the interval $[a(v-1)/b, 1 - v_0 - a - b]$, and $d(u, v_0)$ is monotone decreasing on the interval $[-a, a(v-1)/b]$. Let $1 - v_0 - a - b \geq u_3 > u_2 > u_1 \geq a(v-1)/b$, then $d(u_3, v_0) > d(u_2, v_0) > d(u_1, v_0)$. Thus we have $D(A_3, A_6) > D(A_2, A_5) > D(A_1, A_4)$. Let $a(v-1)/b \geq u_3 > u_2 > u_1 \geq -a$, then $d(u_1, v_0) > d(u_2, v_0) > d(u_3, v_0)$. Thus we have $D(A_1, A_4) > D(A_2, A_5) > D(A_3, A_6)$.

Suppose that $u_3 - u_2 = u_2 - u_1$. For $d(u, v_0)$ is concave, and we get $d''_{uu}(u, v_0) > 0$. It follows that $d'_u(u, v_0)$ is monotone increasing on the interval $[a(v-1)/b, 1 - v_0 - a - b]$. Thus, we have $\frac{d(u_2, v_0) - d(u_1, v_0)}{u_2 - u_1} < \frac{d(u_3, v_0) - d(u_2, v_0)}{u_3 - u_2}$. For $d(u_1, v_0) < d(u_2, v_0) < d(u_3, v_0)$, we have $d(u_2, v_0) - d(u_1, v_0) > 0$, and $d(u_3, v_0) - d(u_2, v_0) > 0$. Therefore, $d(u_2, v_0) - d(u_1, v_0) < d(u_3, v_0) - d(u_2, v_0)$. That is, $D(A_2, A_5) - D(A_1, A_4) < D(A_3, A_6) - D(A_2, A_5)$.

When $-a \leq u < a(v-1)/b$, we also have $d'_u(u, v_0)$ is a monotone increasing function.

It follows that $\frac{d(u_2,v_0)-d(u_1,v_0)}{u_2-u_1} < \frac{d(u_3,v_0)-d(u_2,v_0)}{u_3-u_2}$. For $d(u_1,v_0) > d(u_2,v_0) > d(u_3,v_0)$, we have $d(u_2,v_0)-d(u_1,v_0) < 0$,and $d(u_3,v_0)-d(u_2,v_0) < 0$. Therefore, $d(u_1,v_0)-d(u_2,v_0) > d(u_2,v_0)-d(u_3,v_0)$. That is, $D(A_1,A_4)-D(A_2,A_5) > D(A_2,A_5)-D(A_3,A_6)$.

Example 2. Let us consider two elements $x_A = < x_A, u_A, v_A >$, $x_B = < x_B, u_B, v_B >$ of the IFS B', where $u_B = u_A - 0.1$, $v_B = v_A + 0.2$, $v_A = 0.6$. We denote $u_A = u$ and $v_A = v$. The spherical distance between points A and B are:

$$d(u,0.6) = \arccos((u(u-0.1)^{\frac{1}{2}} + (0.6(0.6+0.2)^{\frac{1}{2}} + ((1-u-0.6)(1-u-0.6+0.1-0.2)^{\frac{1}{2}}), \tag{15}$$

$d(u,0.6)$ is actually an intersecting line between the 3-D surface $d(u,v) = \arccos((u(u-0.1)^{\frac{1}{2}} + (v(v+0.2)^{\frac{1}{2}} + ((1-u-v)(1-u-v+0.1-0.2)^{\frac{1}{2}})$ and 2-D plane $v = 0.6$. (As shown in Fig. 3 and 4). $d(u,0.6)$ is a convex function, and its lowest point is 0.2. $d(u,0.6)$ is decreasing on the interval $[0.1,0.2]$, and it is increasing on the interval $[0.2,0.3]$ (As shown in Fig. 4).

For any $0.3 \ge u_3 > u_2 > u_1 \ge 0.2$, and $u_3 - u_2 = u_2 - u_1$, we have $d(u_3,v_0) > d(u_2,v_0) > d(u_1,v_0)$, and $d(u_2,v_0)-d(u_1,v_0) < d(u_3,v_0)-d(u_2,v_0)$. For any $0.2 \ge u_3 > u_2 > u_1 > 0.1$, and $u_3 - u_2 = u_2 - u_1$, we have $d(u_1,v_0) > d(u_2,v_0) > d(u_3,v_0)$, and $d(u_1,v_0)-d(u_2,v_0) > d(u_2,v_0)-d(u_3,v_0)$. This denotes that $d(u,0.6)$ is decreasing quickly near the left endpoint (0.1) of the interval $[0.1,0.3]$, and $d(u,0.6)$ is increasing quickly near the right endpoint (0.2) of the interval $[0.1,0.3]$ (As shown in Fig. 4).

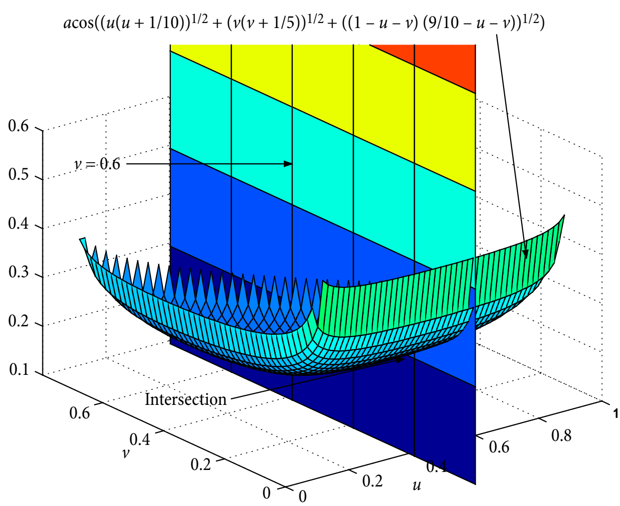

Fig. 3. The intersection between $d(u,v)$ and $v = 0.6$

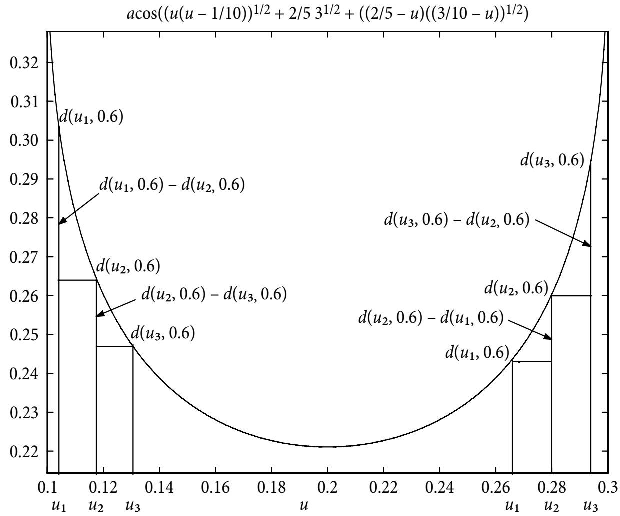

Fig. 4. Figure of function $d(u, 0.6)$

Case 4. $a \geq 0$, $b \leq 0$, and a, b are not both equal to zero.

Subcase 4.1. $a \geq |b|$. For a given number v_0, $v_0 \in [-b, 1-a-b]$. (We can also get the similar conclusion with u_0, $u_0 \in [0, 1-a]$). We only need to consider two intervals $u \in [a(1-a-b-v_0)/(2a+b), 1-v_0-a-b]$ and $u \in [0, a(1-a-b-v_0)/(2a+b)]$, respectively. The conclusion is similar to Subcase 3.1.

Subcase 4.2. $a < |b|$. For a given number v_0, $v_0 \in [-b, 1]$. (We can also get the similar conclusion with u_0, $u_0 \in [0, 1+b]$). We only need to consider two intervals $u \in [a(v_0-1)/b, 1-v_0]$ and $u \in [0, a(v_0-1)/b]$, respectively. The conclusion is similar to Subcase 3.2.

Thus we complete the proof of Property 3 and Property 4.

3. Applications for the spherical distance model of IFS

In literature (Yang, Chiclana 2012), Yang and Chiclana showed that a three dimensional interpretation of intuitionistic fuzzy sets could give different comparison results to the ones obtained with their two dimensional counterparts. They defined a three dimensional Hausdorff distance to show the usefulness of the three dimensional functions to model and provide the expression of the distance between two intuitionistic fuzzy sets. In this section, we will discuss how to apply the spherical distance model to group decision making, pattern recognition, and medical diagnosis.

3.1. The spherical distance model for group decision making

Consider a group decision making situation where group members $d_i, i \in M = \{1, 2, \ldots, m\}$ express their intuitionistic fuzzy opinions $X_i = \{< x_i, \mu_i, \nu_i >\}$, where μ_i and ν_i are the degree of membership and the degree of non-membership of an alternative or an attribute to the fuzzy concept "excellence" given by $d_i, i \in M$, respectively, where $0 \le \mu_i \le 1$, $0 \le \nu_i \le 1$ and $0 \le \mu_i + \nu_i \le 1$. The intuitionistic indices $\pi_i = 1 - \mu_i - \nu_i$ is such that the larger π_i the higher hesitation margin of the decision maker d_i as to the "excellence" of the alternative or the attribute. In intuitionistic fuzzy opinions $X_i = \{< x_i, \mu_i, \nu_i >\}$, $i \in M$, we often call the extreme points $\max_{i \in M} \mu_i$, $\min_{i \in M} \mu_i$, $\max_{i \in M} \nu_i$ and $\min_{i \in M} \nu_i$ the maximum membership degree opinion, the minimum membership degree opinion, the maximum nonmembership degree opinion, and the minimum nonmembership degree opinion, respectively.

In this section, we define three-tuples $R_i = (\mu_i, \nu_i, \pi_i)$ satisfying $0 \le \mu_i \le 1$, $0 \le \nu_i \le 1$ and $\mu_i + \nu_i + \pi_i = 1$ as intuitionistic fuzzy opinions (appraisal values) presented by decision maker (DM) $d_i, i = 1,2,\ldots,m$ with respect to the alternative or the attribute.

Suppose there exists an ideal DM^* whose intuitionistic fuzzy estimation $R^* = (\mu^*, \nu^*, \pi^*)$ is the most desirable one, and $R = (\mu, \nu, \pi)$ is an arbitrary intuitionistic fuzzy estimation. Then, the spherical distance between the ideal estimation and each opinion in group should be smaller than spherical distances between any estimation and each opinions in group. Thus, we construct the following multi-objective nonlinear optimization model, which minimizes the spherical distance between the ideal opinion and each individual opinion in group decision making.

Model 2: (The spherical distance model to get the ideal evaluation of the DM^*)

$$
\begin{aligned}
&mind(R_1, R) = arccos((\mu_1\mu)^{\frac{1}{2}} + (\nu_1\nu)^{\frac{1}{2}} + (\pi_1\pi)^{\frac{1}{2}})(obj_1)\\
&mind(R_2, R) = arccos((\mu_2\mu)^{\frac{1}{2}} + (\nu_2\nu)^{\frac{1}{2}} + (\pi_2\pi)^{\frac{1}{2}})(obj_2)\\
&\cdots\\
&mind(R_m, R) = arccos((\mu_m\mu)^{\frac{1}{2}} + (\nu_m\nu)^{\frac{1}{2}} + (\pi_m\pi)^{\frac{1}{2}})(obj_m)\\
&s.t.\begin{cases} \mu + \nu + \pi = 1 \\ 0 \le \mu \le 1; 0 \le \nu \le 1; 0 \le \pi \le 1. \end{cases}
\end{aligned} \tag{16}
$$

In Model 2, there are m objective functions (goals) obj_i, $i = 1,2,\ldots,m$, and the spherical distance between the ideal opinion R^* and the opinion R_i is smaller than the spherical distance between the opinion R and the opinion $R_i, i \in M$. Generally, it is hard for all these objectives to attain the minimum value simultaneously, so the goals are ranked in order of importance. In general, solving a multi-objective nonlinear optimization problem involves solving a sequence of nonlinear programs with different objective functions, the most important goal is considered firstly, and the least important one is considered lastly. Therefore, we can only get the Pareto optimal solutions instead of the optimal solution of a multi-objective nonlinear optimization Model 2.

In the light of properties of Section 3, the extreme point is vital to decision making. If a group prefers to the maximum membership degree opinion (or the minimum membership degree opinion), the ideal membership degree μ^* should have a smaller spherical distance with the maximum membership degree opinion (or the minimum membership degree opinion); if a group prefers to the maximum nonmembership degree opinion (or the minimum nonmembership degree opinion), the ideal nonmembership degree ν^* should have a smaller spherical distance with the maximum nonmembership degree opinion (or the minimum nonmembership degree opinion). Theses mean that the objective functions with the extreme point (the extreme membership degree opinion or the extreme non-membership degree opinion) should have a high priority. In other words, the minimum value of these functions need to be firstly attained.

Example 3. Let us consider two intuitionistic fuzzy appraisal sets $A'=\{A_2,A_3,A_4\}=\{<A_2,0.5,0.1>,<A_3,0.6,0.1>,<A_4,0.7,0.1>\}$ and $A''=\{A_1,A_2,A_3,A_4,A_5\}=\{<A_1,0.4,0.1>, <A_2,0.5,0.1>,<A_3,0.6,0.1>,<A_4,0.7,0.1>,<A_5,0.8,0.1>\}$. The former is presented by three decision makers $E_1=\{d_2,d_3,d_4\}$, the latter by five decision makers $E_2=\{d_1,d_2,d_3,d_4,d_5\}$. Let us suppose that the group are invited to evaluate the risk of drought, then the extreme opinion $<A_4,0.7,0.1>$ in E_1 and the extreme opinion $<A5,0.8,0.1>$ in E_2 should be paid more attention, respectively.

According to the Property 3 of the spherical distance of IFS, the spherical distance satisfies $D(A_5,A_4)>D(A_4,A_3)>D(A_3,A_2)>D(A_2,A_1)$. It denotes that the membership degree of collective opinion of E_1 and E_2 should go near to the maximum membership degree opinions 0.7 and 0.8, respectively.

According to the Property 4 of the spherical distance of IFS, the spherical distance satisfies $D(A_5,A_4)-D(A_4,A_3)>D(A_4,A_3)-D(A_3,A_2)>D(A_3,A_2)-D(A_2,A_1)$. It denotes that the membership degree of collective opinion of E_2 should be greater than that of E_1.

Suppose that the ideal intuitionistic evaluations in E_1 and E_2 are $R^*_{E_1}=(\mu^*_{E_1},\nu^*_{E_1},\pi^*_{E_1})$, $R^*_{E_2}=(\mu^*_{E_2},\nu^*_{E_2},\pi^*_{E_2})$, respectively, then we construct the following two optimization models to get these two ideal evaluations based on Model 2:

Model 3: (The spherical distance model to get the ideal evaluation of the E_1)

$$\begin{aligned}
&mind(A_1,R)=arccos((0.5\mu)^{\frac{1}{2}}+(0.1\nu)^{\frac{1}{2}}+(0.4\pi)^{\frac{1}{2}})(obj_1)\\
&mind(A_2,R)=arccos((0.6\mu)^{\frac{1}{2}}+(0.1\nu)^{\frac{1}{2}}+(0.3\pi)^{\frac{1}{2}})(obj_2)\\
&mind(A_3,R)=arccos((0.7\mu)^{\frac{1}{2}}+(0.1\nu)^{\frac{1}{2}}+(0.2\pi)^{\frac{1}{2}})(obj_3)\\
&s.t.\begin{cases}\mu+\nu+\pi=1\\ 0\le\mu\le1;0\le\nu\le1;0\le\pi\le1.\end{cases}
\end{aligned}\tag{17}$$

Model 4: (The spherical distance model to get the ideal evaluation of the E_2)

$$\begin{aligned}
&mind(A_1,R)=arccos((0.4\mu)^{\frac{1}{2}}+(0.1\nu)^{\frac{1}{2}}+(0.5\pi)^{\frac{1}{2}})(obj_1)\\
&mind(A_2,R)=arccos((0.5\mu)^{\frac{1}{2}}+(0.1\nu)^{\frac{1}{2}}+(0.4\pi)^{\frac{1}{2}})(obj_2)\\
&mind(A_3,R)=arccos((0.6\mu)^{\frac{1}{2}}+(0.1\nu)^{\frac{1}{2}}+(0.3\pi)^{\frac{1}{2}})(obj_3)\\
&mind(A_4,R)=arccos((0.7\mu)^{\frac{1}{2}}+(0.1\nu)^{\frac{1}{2}}+(0.2\pi)^{\frac{1}{2}})(obj_4)\\
&mind(A_5,R)=arccos((0.8\mu)^{\frac{1}{2}}+(0.1\nu)^{\frac{1}{2}}+(0.1\pi)^{\frac{1}{2}})(obj_5)\\
&s.t.\begin{cases}\mu+\nu+\pi=1\\ 0\le\mu\le1;0\le\nu\le1;0\le\pi\le1.\end{cases}
\end{aligned}\tag{18}$$

Model 3 and Model 4 can be solved by some optimization softwares, such as *the MATLAB Genetic Algorithm Toolbox* (Multi-objective Optimization using Genetic Algorithm). In Model 3, obj_3 is the most important goal, and in Model 4, obj_5 is the most important goal. Therefore, the Pareto optimal solutions to Model 3 and Model 4 are selected as $R_{E_1}^* = (\mu_{E_1}^*, \nu_{E_1}^*, \pi_{E_1}^*) = (0.6720, 0.1058, 0.2230)$, and $R_{E_2}^* = (\mu_{E_2}^*, \nu_{E_2}^*, \pi_{E_2}^*) = (0.78460.09710.1193)$, respectively. Obviously, the membership degree in $R_{E_2}^*$ is greater than that in $R_{E_1}^*$. And we also get $R_{E_2}^* > R_{E_1}^*$ in the light of the method of the comparison between two intuitionistic fuzzy values (Xu 2007).

In the following, we will show the application of the spherical distance in the evaluation of drought risk.

Example 4. Disaster risk analysis is an important and basic research issues related with social economics, mainly including disaster prediction, disaster prevention, disaster related social economic costs, and etc. A natural disaster like drought often has long lasting and far-reaching social, economic and environmental impacts. A reliable risk analysis is essential to have a sustainable social economic development. Usually, expert consultation is important to determine the possibility of disaster risk. The weather bureau invites four experts $E = \{d_1, d_2, d_3, d_4\}$ to evaluate the risk of drought. The intuitionistic fuzzy appraisal values are as follows:

$$< r_1, 0.6, 0.2 >, < r_2, 0.7, 0.1 >, < r_3, 0.4, 0.3 >, < r_4, 0.3, 0.6 > .$$

In the intuitionistic fuzzy appraisal value, the first element is the minimum degree of the assurance that the hazard will occur (the minimum possibility of drought risk), the second value is the degree of the assurance that the hazard will not occur, the third value is the degree of hesitation. Let's take $< r_1, 0.6, 0.2 >$ as an example. The expert d_1 views that the possibility of drought risk is at least 60%, the maximum possibility of drought risk may be 80%, and he or she believes that there is 20% possibility that the drought risk will not occur.

Let $< r_*, \mu_*, \nu_* >$ satisfying $\mu_* + \nu_* \le 1$ denote the intuitionistic fuzzy opinion of an ideal expert r_*, then the spherical distances between r_* and $r_i, i = 1,2,3,4$ are the values such that the spherical distances between the intuitionistic fuzzy opinion $< r, \mu, \nu >$ and $< r_i, \mu_i, \nu_i >, i = 1,2,3,4$ attain the minimum. Hence, we construct the following optimization Model 5:

$$mind(r_1, r) = arccos((0.6\mu)^{\frac{1}{2}} + (0.2\nu)^{\frac{1}{2}} + (0.2\pi)^{\frac{1}{2}})(obj_1)$$

$$mind(r_2, r) = arccos((0.7\mu)^{\frac{1}{2}} + (0.1\nu)^{\frac{1}{2}} + (0.2\pi)^{\frac{1}{2}})(obj_2)$$

$$mind(r_3, r) = arccos((0.4\mu)^{\frac{1}{2}} + (0.3\nu)^{\frac{1}{2}} + (0.3\pi)^{\frac{1}{2}})(obj_3)$$

$$mind(r_4, r) = arccos((0.3\mu)^{\frac{1}{2}} + (0.6\nu)^{\frac{1}{2}} + (0.1\pi)^{\frac{1}{2}})(obj_4)$$

$$s.t. \begin{cases} \mu_* + \nu_* + \pi_* = 1 \\ 0 \le \mu^* \le 1; 0 \le \nu^* \le 1; 0 \le \pi^* \le 1. \end{cases} \tag{19}$$

The opinion of expert d_2 can be viewed as an extreme value, so obj_2 is the most important objective function. We select a Pareto optimal solution to Model (5) as

$\mu_* = 0.6823, \nu_* = 0.1191, \pi_* = 0.1995$. This means that the opinion of the group is $< r_*, 0.6823, 0.1191 >$, and the possibility of drought risk is 68.23% to 88.18%. The result shows the drought risk is considered likely to happen, and the expert d_2 plays an important role.

3.2. Pattern recognition

In order to demonstrate the application of the spherical distance measures for IFSs to pattern recognition, we consider the data proposed by Li and Cheng (2002), Mitchell (2003) and Vlachos and Sergiadis (2007). Consider three known patterns A_1, A_2 and A_3 which have classifications C_1, C_2 and C_3. The patterns are represented by the following IFSs in $X = \{x_1, x_2, x_3\}$

$$A_1 = \{< x_1, 1, 0 >, < x_2, 0.8, 0 >, < x_3, 0.7, 0.1 >\},$$
$$A_2 = \{< x_1, 0.8, 0.1 >, < x_2, 1, 0 >, < x_3, 0.9, 0.0 >\},$$
$$A_3 = \{< x_1, 0.6, 0.2 >, < x_2, 0.8, 0 >, < x_3, 1.0, 0 >\}$$

respectively. Given an unknown pattern B, represented by the IFS $B = \{< x_1, 0.5, 0.3 >, < x_2, 0.6, 0.2 >, < x_3, 0.8, 0.1 >\}$. By using spherical distance measures, we classify B to one of the classes C_1, C_2 and C_3 (As shown in Table 4).

Table 4. Spherical distance measure $D(A_i, B)$ with $i \in \{1, 2, 3\}$

	A_1	A_2	A_3
B	0.2962	0.2830	**0.2232**

where $D(B, A_1) = 0.2962$, $D(B, A_2) = 0.2830$, $D(B, A_3) = 0.2232$. The smaller the spherical distance value is, the bigger similarity between two IFSs. We can observe that B is classified to C_3, and this result is in agreement with the ones obtained in Li and Cheng (2002), Mitchell (2003), and Vlachos and Sergiadis (2007).

3.3. Medical diagnosis

In this subsection, we utilize the new spherical distance measure to show how to carry out medical disgnosis. Let's consider the same data as in (De *et al.* 2001; Szmidt, Kacprzyk 2001a, 2001b, 2002; Vlachos, Sergiadis 2007; Wei *et al.* 2011), consisting of a set of patient $P = \{Al, Bob, Joe, Ted\}$, a set of diagnoses $D = \{Viralfever, Malaria, Typhoid, Stomachproblem, Chestpain\}$, and a set of symptoms $S = \{Temperature, Headache, StomachPain, Cough, Chestpain\}$. Table 5 and Table 6 present the characteristic symptoms of the considered diagnosis, the symptoms of each patient, respectively. Each element of the tables is given in the form of a three-tuple of numbers corresponding the membership degree μ', nonmembership degree ν'_i, and hesitation degree π'_i, respectively. A proper diagnosis for each patient $p_i, i = 1, 2, 3, 4$ needs to be found.

Table 5. Symptoms characteristic for the considered diagnoses

	Viral fever	Malaria	Typhoid	Stomach problem	Chest problem
Temperature	(0.4,0,0.6)	(0.7,0,0.3)	(0.3,0.3,0.4)	(0.1,0.7,0.2)	(0.1,0.8,0.1)
Headache	(0.3,0.5,0.2)	(0.2,0.6,0.2)	(0.6,0.1,0.3)	(0.2,0.4,0.2)	(0,0.8,0.2)
Stomach pain	(0.1,0.7,0.2)	(0,0.9,0.1)	(0.2,0.7,0.1)	(0.8,0,0.2)	(0.2,0.8,0)
Cough	(0.4,0.3,0.3)	(0.7,0,0.3)	(0.2,0.6,0.2)	(0.2,0.7,0.1)	(0.2,0.8,0)
Chest pain	(0.1,0.7,0.2)	(0.1,0.8,0.1)	(0.1,0.9,0)	(0.2,0.7,0.1)	(0.8,0.1,0.1)

Table 6. Symptoms characteristic for the considered patients

	Temperature	Headache	Stomach pain	Cough	Chest pain
Al	(0.8,0.1,0.1)	(0.6,0.1,0.3)	(0.2,0.8,0)	(0.6,0.1,0.3)	(0.1,0.6,0.3)
Bob	(0,0.8,0.2)	(0.4,0.4,0.2)	(0.6,0.1,0.3)	(0.1,0.7,0.2)	(0.1,0.8,0.1)
Joe	(0.8,0.1,0.1)	(0.8,0.1,0.1)	(0,0.6,0.4)	(0.2,0.7,0.1)	(0,0.5,0.5)
Ted	(0.6,0.1,0.3)	(0.5,0.4,0.1)	(0.3,0.4,0.3)	(0.7,0.2,0.1)	(0.3,0.4,0.3)

Table 7. Similarities symptoms (Spherical distance) for each patient to the considered set of possible diagnosis

	Viral fever	Malaria	Typhoid	Stomach problem	Chest problem
Al	**0.2492**	0.2701	0.2566	0.4305	0.4509
Bob	0.3352	0.4675	0.2785	**0.1616**	0.4137
Joe	0.3049	0.3772	**0.2966**	0.4273	0.5361
Ted	**0.2057**	0.2887	0.2905	0.3511	0.4354

The results for the considered patients are given in Table 7. The proper diagnosis d_k for the ith patient p_i is obtained according to the smallest numerical value from the spherical distance measures. These results are in agreement with the ones obtained by Vlachos and Sergiadis (2007).

Conclusions

Euclidean distance may not adequately measure the difference of the qualitative evaluations, for the reason that human perception may sometimes be nonlinear. Yang and Chiclana proposed a simple spherical distance measure and proved that it has good properties. This paper generalized Yang and Chiclana' work (Yang, Chiclana 2009). Firstly, the generalized properties show that the spherical distance measure can simulate the difference of human perception or judgment. Secondly, the spherical distance measure is very suitable for group decision making. Thirdly, the extreme appraisal is also considered adequately. In consequence, we applied the spherical distance measure to get the collective opinion in a group: we first introduced an ideal intuitionistic fuzzy estimation, and then by minimizing the spherical distance between the ideal opinion and each individual opinion in group decision, we constructed a nonlinear optimization model. An optimization model of drought risk analysis based on intuitionistic fuzzy estimation was derived to show the practicality of the proposed method. Additionally, we utilized the same data proposed by Li and Cheng (2002), Mitchell (2003), De *et al.* (2001), Szmidt and Kacprzyk (2001a, 2001b, 2002), Vlachos and Sergiadis (2007) and Wei *et al.* (2011) to show the new spherical distance can also be applied to many fields such as pattern recognition and medical diagnosis.

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