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CONVERGENCE ORDER OF ONE REGULARIZATION METHOD

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ABSTRACT

Modelling many problems of mathematical physics, economy, statistics, actuary mathematics we obtain operational equations of the first kind. As a rule, these equations concern to ill-posed problems. There are some iterative methods for solution of such problems. In the present work we consider the concrete iterative method and estimate its order of convergence without any additional conditions.

Key words: ill-posed problems, iterative methods, numerical algorithms

1. INTRODUCTION

Consider the operator equation of the first kind

$$Az = u, \quad z \in H, \ u \in AH, \tag{1.1}$$

where H is a Hilbert space, the operator $A : H \to H$ is linear, self-conjugate, positive and completely continuous, $u \in AH$ is a given element and $z \in H$ is the unknown element. It is assumed that the zero point does not belong to the spectrum of A and the equation (1.1) is solvable for all $u \in H$, i.e. $AH = H$.

The following two Theorems are known (see [1] and references therein):

Theorem 1.1. *Let* z_p *be an exact solution of* (1.1) *for* $u = u_p \in H$, *i.e.*

$$Az_p = u_p.$$

Then the iterative process

$$z_0 = 0, \quad z_{n+1} = z_n + \theta(u_p - Az_n), \quad n \geq 0 \tag{1.2}$$

converges to the solution of (1.1) *in the norm of the Hilbert space* H *under the condition* $0 < \theta < \dfrac{2}{\|A\|}$.

Theorem 1.2. *Suppose that instead of the exactly given right-hand side* $u = u_p \in H$ *there is some* u_δ *such that* $\|u_p - u_\delta\| \leq \delta$. *Then the iterative process*

$$z_0^\delta = 0, \quad z_{n+1}^\delta = z_n^\delta + \theta(u_\delta - Az_n^\delta), \quad n \geq 0 \tag{1.3}$$

converges to the exact solution z_p *of the equation* (1.1) *in the norm of the Hilbert space* H *and the following inequality holds*

$$\|z_n^\delta - z_p\| \leq \|z_n - z_p\| + \varepsilon(n)\delta,$$

where

$$\varepsilon(n) = \theta \sum_{k=0}^{n-1} \|E - \theta A\|^k,$$

E *is the identity operator and* z_n *is given by* (1.2).

Using Theorem 1.1 and choosing $n = n(\delta)$ such that $\varepsilon(n(\delta))\delta \to 0$ as $\delta \to 0$ we obtain from (1.3) that z_n^δ converges to z_p.

Thus, $z_n^\delta \to z_p$ in the norm of the space H as $n(\delta) \to \infty$, but the order of convergence can be arbitrary small. In order to estimate the order of convergence one needs to assume that the exact solution (which is not known) is source-wise representable. The following theorem is proved in [2]:

Theorem 1.3. *Suppose that the exact solution* z_p *of equation* (1.1) *is source-wise representable, i.e.*

$$z_p = A^\sigma s, \quad \sigma > 0.$$

Then the following inequality holds

$$\|z_p - z_n^\delta\| \leq \sigma^\sigma (n\theta e)^{-\sigma} \|s\| + n\theta\delta, \tag{1.4}$$

where $0 < \theta < \dfrac{2}{\|A\|}$. *Here* z_n^δ *is calculated from* (1.3).

2. FAILURE ON THE SOURCE-WISE REPRESENTABILITY CONDITION OF THE EXACT SOLUTION

We will prove that in order to estimate the order of convergence of iterative process (1.3) the strong additional condition on source-wise representability of the exact solution is not required. Let us use the energy norm, then the following Theorem holds for the iterative process (1.3) under the condition $0 < \theta < \dfrac{2}{3\|A\|}$.

Theorem 2.1. *The iterative process* (1.3) *is convergent in the energy norm if the number of iterations n is chosen from the condition*

$$\sqrt{n}\delta \underset{n\to\infty,\delta\to 0}{\longrightarrow} 0.$$

In addition, under the condition

$$0 < \theta \le \frac{4}{3\|A\|} \tag{2.1}$$

the following error estimates are valid for iterative process (1.3)

$$\begin{aligned} \|z_p - z_n^\delta\| &\le (2n\theta e)^{-\frac{1}{2}}\|z_p\|_A + \Big(\frac{4}{3}n\theta\Big)^{\frac{1}{2}}\delta, \quad n \ge 1, \\ \|z_p - z_n^\delta\| &\le (2n\theta e)^{-\frac{1}{2}}\|z_p\|_A + \Big(\frac{13}{20}n\theta\Big)^{\frac{1}{2}}\delta, \quad n \ge 2. \end{aligned} \tag{2.2}$$

Proof. Let us transform (1.2) to the form

$$z_n = z_{n-1} + \theta(u_p - Az_{z-1}) = \big(E - (E-\theta A)^n\big)z_p, \quad n \ge 1.$$

Similarly, from (1.3) we get

$$\begin{aligned} z_n^\delta &= \theta\sum_{k=0}^{n-1}(E-\theta A)^k u_\delta, \quad n \ge 1, \\ z_p - z_n^\delta &= \big(z_p - z_n\big) + \big(z_n - z_n^\delta\big), \\ z_p - z_n &= \big(E-\theta A\big)^n z_p. \end{aligned}$$

Since it is assumed that the operator A is self-conjugate, one has $A = \int\limits_m^M \lambda dE_\lambda$, where

$$m = \inf_{\|x\|=1}(Ax,x) > 0, \; M = \sup_{\|x\|=1}(Ax,x) > 0,$$

and E_λ is a projective operator. Using the formula

$$\|f(A)\|_A^2 = (Af(A)x, f(A)x) = \int_m^M \lambda f^2(\lambda) d(E_\lambda x, x),$$

we obtain that

$$\|z_p - z_n\|_A^2 = \Big(A\big(E - \theta A\big)^n z_p, (E - \theta A)^n z_p\Big) = \int_m^M \lambda(1 - \theta\lambda)^{2n} d(E_\lambda z_p, z_p).$$

We define the function $f(\lambda)$ by the formula

$$f(\lambda) \overset{\text{def}}{\equiv} \lambda(1 - \theta\lambda)^{2n}.$$

In order to find the upper bound of $\|z_p - z_n\|_A^2$ one needs to maximize the function $f(\lambda)$ in the closed interval $\big[m, M\big]$, i.e. $f(\lambda) \longrightarrow \max\limits_{\lambda\in[m,M]}$. Using the necessary condition for the existence of an extremum $f'(\lambda) = 0$ we obtain a stationary point

$$\lambda^* = \frac{1}{\theta(2n+1)}.$$

It is known that if $f''(\lambda) < 0$ then $\lambda = \lambda^*$ is a point of local maximum. It can be verified that $f''(\lambda^*) < 0$.

Thus, the function $f(\lambda)$ has a local maximum at $\lambda = \lambda^*$. The upper bound of $f(\lambda^*)$ is

$$f(\lambda^*) = \frac{1}{2n\theta}\Big(\frac{2n}{2n+1}\Big)^{2n+1} = \frac{1}{2n\theta}\Big[\Big(1 + \frac{1}{2n}\Big)^{2n}\Big]^{-\left(1+\frac{1}{2n}\right)} < \frac{1}{2n\theta e}.$$

It can be shown that $\max\limits_{\lambda\in[m,M]} f(\lambda) < \dfrac{1}{2n\theta e}$ if $0 < \theta \le \dfrac{4}{3\|A\|}$. Indeed, $f(\|A\|) = \|A\|\big(1 - \theta\|A\|\big)^{2n}$ and the function $f(\lambda)$ has a maximum at

$$\lambda^* = \frac{1}{\theta(2n+1)}$$

under the condition $\theta\lambda < 1$.

If $\theta\lambda > 1$ then the larger is θ, the greater is $f(\|A\|)$. Therefore it is sufficient to calculate $f(\|A\|)$ at $\theta = \dfrac{4}{3\|A\|}$. Thus,

$$f(\|A\|)_{\theta=\frac{4}{3\|A\|}} = \frac{\|A\|}{9^n}.$$

Let us prove the inequality:

$$f\Big(\|A\|\Big)\Big|_{\theta=\frac{4}{3\|A\|}} < \frac{1}{2n\theta e}\Big|_{\theta<\frac{4}{3\|A\|}}.$$

This inequality is equivalent to the inequality

$$8ne < 3\cdot 9^n, \quad n \geq 1.$$

Therefore, for $\theta \in \left(0, \dfrac{4}{3\|A\|}\right]$ one has

$$\max_{\lambda\in[m,M]} f(\lambda) < \frac{1}{2n\theta e}.$$

Thus $\|z_p - z_n\|_A^2 \leq (2n\theta e)^{-1}\|z_p\|_A^2$, so that $\|z_p - z_n\|_A \leq (2n\theta e)^{-1/2}\|z_p\|_A$. The upper bound for $\|z_p - z_n^\delta\|_A^2$ is obtained from

$$\begin{aligned}
z_n - z_n^\delta &= \theta\sum_{k=0}^{n-1}(E-\theta A)^k(u_p - u_\delta),\\
\|z_n - z_n^\delta\|_A^2 &= \int_m^M \lambda\Big[\theta\sum_{k=0}^{n-1}(1-\theta\lambda)^k\Big]^2 d\big(E_\lambda(u_p-u_\delta), u_p - u_\delta\big) =\\
&= \int_m^M \lambda^{-1}\Big(1-(1-\theta\lambda)^n\Big]^2 d\big(E_\lambda(u_p-u_\delta), u_p-u_\delta\big).
\end{aligned}$$

Let us introduce the notation $g_n(\lambda) = \lambda^{-1}(1-(1-\theta\lambda)^n)^2$. Now we obtain the upper bound for $g_n(\lambda)$ under the condition $0 < \theta \leq \dfrac{4}{3\|A\|}$. For $n = 1$ we have $g_1(\lambda) = \Theta^2\lambda \leq \frac{4}{3}\theta$. For $n = 2$ we have $g_2(\lambda) = \theta^2\lambda(2-\theta\lambda)^2$. We must maximize the function $g_2(\lambda)$ in the closed interval $[m, M]$, i.e. $g_2(\lambda) \longrightarrow \max\limits_{\lambda\in[m,M]}$. Hence $\lambda^{**} = \dfrac{2}{3\theta}$ is a stationary point. Since $g_2''(\lambda^{**}) = -4\theta^3 < 0$, then λ^{**} is the point of a local maximum and $g_2(\lambda^{**}) = \frac{32}{27}\theta$. It can be shown by induction that $g_1(\lambda) \leq \frac{4}{3}\theta, \quad g_n(\lambda) \leq \frac{13}{20}n\theta, \quad n \geq 2$. Thus,

$$\begin{aligned}
\|z_p - z_n^\delta\|_A &\leq \Big(\frac{13}{20}n\theta\Big)^{1/2}\delta \quad \text{if} \quad n \geq 2,\\
\|z_p - z_n^\delta\|_A &\leq \Big(\frac{4}{3}n\theta\Big)^{1/2}\delta \quad \text{if} \quad n \geq 1.
\end{aligned}$$

Moreover, it follows from the inequality

$$\|z_p - z_n^\delta\|_A \leq \|z_p - z_n\|_A + \|z_n - z_n^\delta\|_A$$

that

$$\|z_p - z_n^\delta\|_A \leq (2n\theta e)^{-\frac{1}{2}}\|z_p\|_A + \Big(\frac{4}{3}n\theta\Big)^{\frac{1}{2}}\delta, \quad n \geq 1,$$
$$\|z_p - z_n^\delta\|_A \leq (2n\theta e)^{-\frac{1}{2}}\|z_p\|_A + \Big(\frac{13}{20}n\theta\Big)^{\frac{1}{2}}\delta, \quad n \geq 2.$$

It can be seen from the above inequalities that the first term tends to zero as $n \to \infty$, and it is sufficient to require $\sqrt{n}\delta \to 0$ as $n \to \infty, \delta \to 0$ in order to get

$$\|z_p - z_n^\delta\|_A \underset{n\to\infty,\delta\to 0}{\longrightarrow} 0.$$

Theorem 2.1 is proved. ■

3. DETERMINATION OF THE OPTIMAL NUMBER OF ITERATIONS

Let us find such $n = n(\delta)$, depending on the given error δ of the right-hand side of (1.1), for which the estimates (2.2) are minimal. Differentiating the right-hand side of (2.2) and equating it to zero we obtain

$$-\frac{1}{2}(2\theta n)^{-\frac{1}{2}}n^{-\frac{3}{2}}\|z_p\|_A + \frac{1}{2}\Big(\frac{13}{20}\theta\Big)^{\frac{1}{2}}\delta n^{-\frac{1}{2}} = 0.$$

Thus,

$$n_{optim} = \Big(\frac{13}{10}\Big)^{-\frac{1}{2}}\theta^{-1}e^{-\frac{1}{2}}\delta^{-1}\|z_p\|_A. \tag{3.1}$$

Substituting (3.1) into (2.2) we obtain

$$\|z_p - z_n^\delta\|_A^{optim} \leq \left(\frac{13}{10}\right)^{\frac{1}{4}}(2\delta\|z_p\|_A)^{\frac{1}{2}}. \tag{3.2}$$

It is seen from (3.2) that the optimal error estimate does not depend on the iteration parameter θ. However, as can be seen from (3.1), the optimal number of iterations n_{optim} depends on θ. Therefore, in order to reduce the number of iterations, i.e. to reduce calculation cost, one needs to choose θ as large as possible from the condition $0 < \theta \leq \frac{4}{3\|A\|}$ so that n_{optim} would be an integer, for example,

$$n_{optim} = \Big[0,4\Big(\frac{\|u_\delta\|}{\delta} - 1\Big)\Big],$$

here $[t]$ denotes the integer part of t. Thus, we have proved the following theorem:

Theorem 3.1. *The optimal error estimate for iterative process* (1.3) *under the conditions of Theorem 2.1 has the form*

$$\|z_p - z_n^\delta\|_A^{optim} \le \left(\frac{13}{10}\right)^{\frac{1}{4}} \left(2\delta\|z_p\|_A\right)^{\frac{1}{2}}$$

and this bound is reached for

$$n_{optim} = \left[\sqrt{\frac{10}{13e}}\frac{1}{\theta\delta}\|z_p\|_A\right].$$

4. THE ESTIMATION OF CONVERGENCE ORDER IN THE INITIAL HILBERT SPACE NORM

Let us show that the results obtained above also hold in the of initial Hilbert space H norm. Indeed, since

$$\begin{aligned}\|z_p - z_n^\delta\|_A = \Big(A\big(z_p - z_n^\delta\big), z_p - z_n^\delta\Big) &\le M\left(z_p - z_n^\delta, z_p - z_n^\delta\right) \\ &= M\|z_p - z_n^\delta\|_H^2, \\ \|z_p - z_n^\delta\|_A = \Big(A\big(z_p - z_n^\delta\big), z_p - z_n^\delta\Big) &\le m\big(z_p - z_n^\delta, z_p - z_n^\delta\big) \\ &= m\|z_p - z_n^\delta\|_H^2,\end{aligned}$$

where $M = \sup\limits_{\|x\|=1} (Ax, x) > 0,\ m = \inf\limits_{\|x\|=1} (Ax, x) > 0$, then

$$\sqrt{m}\big\|z_p - z_n^\delta\big\|_H \le \big\|z_p - z_n^\delta\big\|_A \le \sqrt{M}\big\|z_p - z_n^\delta\big\|_H. \tag{4.1}$$

Two-sided inequality (4.1) allows one to assert that the sequence $\|z_p - z_n^\delta\|_H$ converges to zero if and only if the sequence $\|z_p - z_n^\delta\|_A$ converges to zero. Since the convergence of $\|z_p - z_n^\delta\|_A$ is proved in Theorem 2.1, the following theorem holds:

Theorem 4.1. *The iterative process* (1.3) *converges in the norm of the given Hilbert space* H, *if the number of iterations* n *is chosen from the condition* $\sqrt{n}\delta \to 0$ *as* $n \to \infty, \delta \to 0$. *Moreover, under the condition* $0 < \theta \le \frac{4}{3\|A\|}$ *the following error estimates hold for the iterative process* (1.3)

$$\begin{aligned}\|z_p - z_n^\delta\|_H &\le (2nm\theta e)^{-\frac{1}{2}}\|z_p\|_H + \left(\frac{4}{3}\frac{n}{m}\theta\right)^{\frac{1}{2}}\delta,\ n \ge 1, \\ \|z_p - z_n^\delta\|_H &\le (2nm\theta e)^{-\frac{1}{2}}\|z_p\|_H + \left(\frac{13}{20}\frac{n}{m}\theta\right)^{\frac{1}{2}}\delta,\ n \ge 2,\end{aligned} \tag{4.2}$$

where $m = \inf\limits_{\|x\|=1} (Ax, x) > 0$.

Unlike to Theorem 1.3, in Theorem 4.1 a strong additional condition on source-wise representability of the exact solution of equation (1.1) is not required, but we narrow 1.5 times the domain of definition of the iteration parameter θ. However, the obtained estimates (4.2) show that for fixed δ and θ iterative process (1.3) converges to the exact solution z_p of equation (1.1) with order $\sqrt{n}$, i.e. we have improved the result of Theorem 1.3.

Remark 4.1. The operator A is assumed to be self-conjugate and positive. If the operator A is not self-conjugate or is not positive, then equivalent equation

$$A^*Az = A^*u, \quad z \in H, \quad u \in AH, \tag{4.3}$$

should be used instead of equation (1.1), where the operator A^* is conjugate to A. All the results obtained above are valid in this case.

Remark 4.2. All the results obtained above take place if zero does not belong to the spectrum of the operator A. If zero point belongs to the spectrum of the operator A, then equation (1.1) (or equation (4.3)), can have infinitely many solutions. The approach described above and all the results obtained in the paper are valid also in this case. The method described above guarantees convergence to the normal solution, i.e. to the solution with minimal norm [1].

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