Mathematical Modelling and Analysis
Volume 22 Number 2, March 2017, 213–227
https://doi.org/10.3846/13926292.2017.1291456

Publisher: Taylor&Francis and VGTU
http://www.tandfonline.com/TMMA
ISSN: 1392-6292
eISSN: 1648-3510

Green's Matrices for First Order Differential Systems with Nonlocal Conditions*

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Received November 23, 2016; revised February 1,2017; published online March 1, 2017

Abstract. In this paper, we investigate the linear system of first order ordinary differential equations with nonlocal conditions. Green's matrices, their explicit representations and properties are considered as well. We present the relation between the Green's matrix for the system and the Green's function for the differential equation. Several examples are also given.

Keywords: differential problem, nonlocal conditions, Green's matrix, Green's function.

AMS Subject Classification: 34B10; 34B27; 34L99.

1 Introduction

In 1828 George Green wrote an article "An Essay on the Application of Mathematical Analysis to the Theories of Elasticity and Magnetism", where he investigated a potential, which was later called a *Green's function*. Classically, a Green's function is understood as a kernel, which represents a solution to the differential problem of mathematical physics. However, the modern concept of a Green's function was introduced a little bit later by B. Riemann in [11].

Nowadays, the classical meaning of a Green's function is widely extended, first, to a *generalized Green's function*, which may represent some optimization solution not only to a differential problem [8,9]. Theories of numerical methods and discrete structures also talk about a *discrete Green's function* and it's applications [5, 6, 10]. Many works discuss about *Green's matrices*, those arise investigating algebraic or differential systems.

* The research was partially supported by the Research Council of Lithuania (grant No. MIP-047/2014).

In example, Brown [1] investigated Green's matrices for the differential system

$$\mathbf{u}' = \mathbf{A}\mathbf{u} + \mathbf{f}, \tag{1.1}$$

$$\int_0^1 d\mathbf{F}\,\mathbf{u} = \mathbf{0}, \tag{1.2}$$

where $\mathbf{u}$ is a n-dimensional absolutely continuous vector valued function, $\mathbf{A}$ is a $n \times n$ continuous matrix on $[0,1]$ and $\mathbf{F}$ is a $m \times n$ matrix valued measure of bounded variation elementwise. Green's matrices and their properties for very relative problems were also investigated by other authors [3, 7, 17]. Moreover, Brown and Krall considered an adjoint problem and presented an eigenvalue expansion of a Green's matrix [2].

According to the Riesz representation theorem, Stieltjes conditions (1.2) describe general linear conditions for continuous $\mathbf{u}$. Nowadays they are called *nonlocal conditions* [15] since link function at several different points or the whole interval. Most nonlocal conditions may be of the form

$$\sum_{i=1}^{\infty} \mathbf{A}_i \mathbf{u}(\xi_i) + \int_0^1 \mathbf{B}(x)\mathbf{u}(x)\,dx = 0, \tag{1.3}$$

whereas classical initial conditions are local because they link function or it's derivative at the one point of the interval.

In this paper, we also investigate differential systems (1.1)–(1.2) with nonlocal conditions of the form (1.3) developing relative results, obtained for n-th order differential equations with nonlocal conditions. Indeed, we obtain representations of the unique solution to the system, those are analogues for the differential equation. Presented properties of the Green's matrix and it's expression also literally resemble known properties and existence conditions of a Green's function for the equation, derived by Roman and Štikonas [12,13,14,16]. In the last section, we consider the relation between a n-th order ordinary differential equation and a differential system with nonlocal conditions, where the representation of a Green's function via a Green's matrix and vice versa are derived.

2 System with nonlocal conditions

Let us consider the linear system of ordinary differential equations with nonlocal conditions

$$\frac{d\,u_i}{dx} = \sum_{j=1}^{n} a_{ij}(x)u_j + f_i(x), \quad i = \overline{1,n}, \quad x \in [0,1],$$

$$\sum_{i=1}^{n} \langle L_{ki}, u_i \rangle = g_k, \quad k = \overline{1,n},$$

where $u_i \in C^1[0,1], f_i \in C[0,1],\ g_k \in \mathbb{R},\ L_{ki} \in C^*[0,1],\ i,k = \overline{1,n},\ a_{ij} \in C[0,1]$. Introducing notations $\mathbf{u} = (u_1, u_2, \ldots, u_n)^\top$, $\mathbf{u}' = ({u_1}', {u_2}', \ldots, {u_n}')^\top$,

$\mathbf{A} := \mathbf{A}(x) = (a_{ij}(x))$, $\mathbf{f} := \mathbf{f}(x) = (f_1(x), \dots, f_n(x))^\top$, $x \in [0,1]$, $\mathbf{L}_k = (L_{k1}, \dots, L_{kn})$, the system can be written in the vectorial form

$$\mathbf{u}' = \mathbf{A}\mathbf{u} + \mathbf{f}, \tag{2.1}$$

$$\langle \mathbf{L}_k, \mathbf{u} \rangle = g_k, \quad k = \overline{1,n}. \tag{2.2}$$

Here $\langle \mathbf{L}_k, \mathbf{u} \rangle$ is the usual matrix multiplication $\mathbf{L}_k\mathbf{u}$, where brackets emphasize only the nature of nonlocal conditions. Similarly, $\langle \mathbf{L}_k, \mathbf{U} \rangle = \mathbf{L}_k\mathbf{U}$ for every $n \times m$ matrix $\mathbf{U}$ on $[0,1]$.

2.1 Existence of the unique solution

Let us consider the condition of the existence of the unique solution to the system (2.1)–(2.2). This condition was represented by other authors [1, 3]. However, developing the parallel to n-th order differential equations with nonlocal conditions, we present here this condition in the similar form as an existence condition for the unique solution to the n-th order equation is given in [13].

So, we investigate the existence of the unique trivial solution $\mathbf{z} \equiv \mathbf{0}$ to the homogenous system

$$\mathbf{z}' = \mathbf{A}\mathbf{z}, \tag{2.3}$$

$$\langle \mathbf{L}_k, \mathbf{z} \rangle = 0, \quad k = \overline{1,n}. \tag{2.4}$$

The fundamental system of the equation (2.3) is composed of n linearly independent vectorial functions $\mathbf{z}^j$, $j = \overline{1,n}$, those can be represented by the $n \times n$ order fundamental matrix $\mathbf{Z}(x) = (\mathbf{z}^1(x), \dots, \mathbf{z}^n(x))$. Then we put the general solution into (2.3), which is $\mathbf{z} = \sum_{j=1}^n c_j\mathbf{z}^j$ for $c_j \in \mathbb{R}$, $j = \overline{1,n}$, into conditions (2.4) and obtain the system of n homogenous equations with unknowns c_j, $j = \overline{1,n}$, as follows

$$\sum_{j=1}^n \langle \mathbf{L}_k, \mathbf{z}^j \rangle c_j = 0, \quad k = \overline{1,n}.$$

So, the system has only the trivial solution $\mathbf{z} = \mathbf{0}$ or equivalently $c_j = 0$, $j = \overline{1,n}$, if and only if the determinant of the previous system is nonzero

$$\begin{vmatrix} \langle \mathbf{L}_1, \mathbf{z}^1 \rangle & \langle \mathbf{L}_1, \mathbf{z}^2 \rangle & \dots & \langle \mathbf{L}_1, \mathbf{z}^n \rangle \\ \langle \mathbf{L}_2, \mathbf{z}^1 \rangle & \langle \mathbf{L}_2, \mathbf{z}^2 \rangle & \dots & \langle \mathbf{L}_2, \mathbf{z}^n \rangle \\ \dots & \dots & \dots & \dots \\ \langle \mathbf{L}_n, \mathbf{z}^1 \rangle & \langle \mathbf{L}_n, \mathbf{z}^2 \rangle & \dots & \langle \mathbf{L}_n, \mathbf{z}^n \rangle \end{vmatrix} \neq 0. \tag{2.5}$$

It literally resembles the existence condition of the unique solution to a n-th order differential equation with n nonlocal conditions [13].

3 Representation of the solution

3.1 General solution to the nonhomogenous equation

By [4], the general solution to the equation (2.1) is of the form

$$\mathbf{u}(x) = \sum_{k=1}^n c_k\mathbf{z}^k(x) + \int_0^x \mathbf{K}(x,y)\mathbf{f}(y)\,dy,$$

where $c_k \in \mathbb{R}$, $k = \overline{1,n}$, and $\mathbf{K}(x,y) = \mathbf{Z}(x)\mathbf{Z}^{-1}(y)$ is *the Cauchy matrix*. Moreover, the function $\mathbf{u}^{\mathrm{c}}(x) = \int_0^x \mathbf{K}(x,y)\mathbf{f}(y)\,dy$ is the unique solution to *the Cauchy problem* (2.1), $u_k(0) = 0$, $k = \overline{1,n}$, and always exists. Using the Cauchy matrix, we introduce the Green's matrix for the mentioned Cauchy problem as follows

$$\mathbf{G}^{\mathrm{c}}(x,y) = \begin{cases} \mathbf{K}(x,y), & y \le x, \\ 0, & y > x. \end{cases} \tag{3.1}$$

If the condition (2.5) is satisfied, then the problem (2.1)–(2.2) has the unique exact solution with every right hand side. Thus, we take the particular fundamental system $\mathbf{v}^j$, $j = \overline{1,n}$, which is biorthogonal with respect to the functionals $\mathbf{L}_k$, $k = \overline{1,n}$, i.e., $\langle \mathbf{L}_k, \mathbf{v}^j \rangle = \delta_k^j$, $j,k = \overline{1,n}$. Here δ_k^j is the Kronecker delta. Then the unique solution to the problem (2.1)–(2.2) can also be represented by

$$\mathbf{u}(x) = \sum_{k=1}^{n} c_k \mathbf{v}^k(x) + \int_0^1 \mathbf{G}^{\mathrm{c}}(x,y)\mathbf{f}(y)\,dy. \tag{3.2}$$

3.2 The unique solution

We obtain the unique solution to the problem (2.1)–(2.2) choosing particular values of constants c_k, $k = \overline{1,n}$. Thus, we put the general solution (3.2) of the equation (2.1) into conditions (2.2). Remembering that functionals $\mathbf{L}_k$, $k = \overline{1,n}$, are linear and functions $\mathbf{v}^j$, $j = \overline{1,n}$, are biorthogonal, we obtain

$$c_k = g_k - \Big\langle \mathbf{L}_k, \int_0^1 \mathbf{G}^{\mathrm{c}}(\cdot,y)\mathbf{f}(y)\,dy \Big\rangle, \quad k = \overline{1,n}, \tag{3.3}$$

or

$$c_k = g_k - \langle \mathbf{L}_k, \mathbf{u}^{\mathrm{c}} \rangle, \quad k = \overline{1,n}.$$

Now we put these values of constants into the formula (3.2) and get the representation of the unique solution to the problem (2.1)–(2.2) via the unique solution $\mathbf{u}^{\mathrm{c}}$ to the Cauchy problem as below

$$\mathbf{u} = \sum_{k=1}^{n} (g_k - \langle \mathbf{L}_k, \mathbf{u}^{\mathrm{c}} \rangle)\mathbf{v}^k + \mathbf{u}^{\mathrm{c}}. \tag{3.4}$$

3.3 Relation between solutions

Let us now consider two systems with the same equation but different nonlocal conditions

$$\begin{aligned} &\mathbf{u}' = \mathbf{A}\mathbf{u} + \mathbf{f}, && \mathbf{v}' = \mathbf{A}\mathbf{v} + \mathbf{f}, \\ &\langle \mathbf{l}_k, \mathbf{u} \rangle = \widetilde{g}_k, \quad k = \overline{1,n}, && \langle \mathbf{L}_k, \mathbf{v} \rangle = g_k, \quad k = \overline{1,n}, \end{aligned} \tag{3.5}$$

supposing these problems have unique solutions $\mathbf{u}$ and $\mathbf{v}$, respectively. Their difference $\mathbf{w} = \mathbf{v} - \mathbf{u}$ is the unique solution to the problem

$$\mathbf{w}' = \mathbf{A}\mathbf{w}, \qquad \langle \mathbf{L}_k, \mathbf{w} \rangle = g_k - \langle \mathbf{L}_k, \mathbf{u} \rangle, \quad k = \overline{1,n}.$$

For this problem, we apply the formula (3.4), which represents the unique solution to (2.1)–(2.2). Now $\mathbf{u}^c = \mathbf{0}$ and right hand sides g_k are replaced by $g_k - \langle \mathbf{L}_k, \mathbf{u}\rangle$ for $k = \overline{1,n}$. So, we obtain

$$\mathbf{w} = \sum_{k=1}^{n} (g_k - \langle \mathbf{L}_k, \mathbf{u}\rangle)\mathbf{v}^k.$$

Lemma 1. *The relation between the solutions of the problems* (3.5)

$$\mathbf{v} = \mathbf{u} + \sum_{k=1}^{n} (g_k - \langle \mathbf{L}_k, \mathbf{u}\rangle)\mathbf{v}^k \tag{3.6}$$

is valid, where $\mathbf{v}^j$, $j = \overline{1,n}$, *are the biorthogonal fundamental system of the second problem* (3.5).

3.4 Relation between two biorthogonal fundamental systems

Let us take biorthogonal fundamental systems $\tilde{\mathbf{v}}^j$, $j = \overline{1,n}$, and $\mathbf{v}^j$, $j = \overline{1,n}$, for problems (3.5), respectively. These functions are unique solutions to problems (3.5), where $\mathbf{f} = \mathbf{0}$ and $\tilde{g}_k = g_k = \delta_k^j$, $k = \overline{1,n}$. For every fixed $j = \overline{1,n}$, we apply the formula (3.6) taking $\mathbf{v} = \mathbf{v}^j$ and $\mathbf{u} = \tilde{\mathbf{v}}^j$, i.e.

$$\mathbf{v}^j = \tilde{\mathbf{v}}^j + \sum_{k=1}^{n} (\delta_k^j - \langle \mathbf{L}_k, \tilde{\mathbf{v}}^j\rangle)\mathbf{v}^k, \qquad j = \overline{1,n}.$$

Rewriting, we obtain the linear system

$$\sum_{k=1}^{n} \langle \mathbf{L}_k, \tilde{\mathbf{v}}^j\rangle \mathbf{v}^k = \tilde{\mathbf{v}}^j, \qquad j = \overline{1,n}$$

with the nonsingular matrix $(\langle \mathbf{L}_k, \tilde{\mathbf{v}}^j\rangle)$, $k, j = \overline{1,n}$, since (2.5) is valid for every fundamental system and, particulary, for $\mathbf{z}^j = \tilde{\mathbf{v}}^j$, $j = \overline{1,n}$.

Lemma 2. *The relation*

$$\sum_{k=1}^{n} \langle \mathbf{L}_k, \tilde{\mathbf{v}}^j\rangle \mathbf{v}^k = \tilde{\mathbf{v}}^j, \qquad j = \overline{1,n}$$

between the fundamental systems for the problems (3.5) *is valid.*

The obtained relation between two biorthogonal fundamental systems allows us to find the biorthogonal fundamental system if the biorthogonal fundamental system for other relative problem is known.

4 The Green's matrix

4.1 Representation of the Green's matrix

Let us now investigate the problem (2.1)–(2.2) with homogenous nonlocal conditions, i.e. $g_k = 0$, $k = \overline{1,n}$. As in [1], we rewrite the formula (3.3) in the form

$$c_k = -\int_0^1 \langle \mathbf{L}_k, \mathbf{G}^{\mathrm{c}}(\cdot, y)\rangle \mathbf{f}(y)\, dy, \quad k = \overline{1, n}.$$

Putting these expressions into (3.2), we obtain the following representation of the solution

$$\mathbf{u}(x) = \int_0^1 \big(\mathbf{G}^{\mathrm{c}}(x, y) - \sum_{k=1}^{n} \mathbf{v}^k(x)\langle \mathbf{L}_k, \mathbf{G}^{\mathrm{c}}(\cdot, y)\rangle\big)\mathbf{f}(y)\, dy$$

or simply

$$\mathbf{u}(x) = \int_0^1 \mathbf{G}(x, y)\mathbf{f}(y)\, dy.$$

Here we denoted the kernel

$$\mathbf{G}(x, y) = \mathbf{G}^{\mathrm{c}}(x, y) - \sum_{k=1}^{n} \mathbf{v}^k(x)\langle \mathbf{L}_k, \mathbf{G}^{\mathrm{c}}(\cdot, y)\rangle, \tag{4.1}$$

which is called *the Green's matrix for the system with nonlocal conditions* (2.1)–(2.2).

4.2 Properties of the Green's matrix for the nonlocal problem

First, every condition (2.2) can also be represented by

$$\langle \mathbf{L}_k, \mathbf{u}\rangle = \int_0^1 d\mathbf{F}_k\, \mathbf{u}, \quad k = \overline{1, n},$$

where $\mathbf{F}_k$ is a $1 \times n$ row matrix of bounded variation elementwise. Since considering conditions are of the form (1.3), here $\mathbf{F}_k$ can have at most countably many discontinuities $y_l,\ l = 1, 2, 3, \dots$, and need only be differentiable almost everywhere, i.e. $\mathbf{F}_k' = \mathbf{L}_k$ almost everywhere. Thus, the Green's matrix (4.1) may have at most countably many discontinuities $y_l,\ l = 1, 2, 3, \dots$, as well [3]. In other words, the square $[0, 1] \times [0, 1]$ may be divided into $N \in \overline{\mathbb{N}}$ rectangular domains (each rectangle $x \in [0, 1],\ y_{l-1} < y < y_l,\ l = \overline{1, N},\ y_0 = 0,\ y_N = 1$), where the Green's matrix (4.1) may satisfy the analogue of classical properties of the Green's matrix (3.1) or the Green's function [13]. Indeed, properties of the Green's matrix for the problem with nonlocal conditions, except the discontinuities, were obtained by other authors [1,3], where nonlocal conditions where often called *general conditions*. Below we list several properties, those resemble the classical results for the Green's function, or can be obtained applying the properties of (3.1) to (4.1). So, except the discontinuities $y = y_0,\ y_1,\ y_2, \dots$, we have the following properties of the Green's matrix (4.1):

1) $\mathbf{G}(y + 0, y) = \mathbf{I} - \sum_{k=1}^{n} \mathbf{v}^k(y)\langle \mathbf{L}_k, \mathbf{G}^{\mathrm{c}}(\cdot, y)\rangle$,
   $\mathbf{G}(y - 0, y) = -\sum_{k=1}^{n} \mathbf{v}^k(y)\langle \mathbf{L}_k, \mathbf{G}^{\mathrm{c}}(\cdot, y)\rangle$;
2) $\mathbf{G}(y + 0, y) - \mathbf{G}(y - 0, y) = \mathbf{I}$;
3) $\mathbf{G}(x, y)$ is C in (x, y) except the diagonal $x = y$;

5) $\mathbf{G}(x,y)$ is C^1 in x except the diagonal $x=y$;

6) $(\partial/\partial x)\mathbf{G}(x,y) - \mathbf{A}(x)\mathbf{G}(x,y) = \mathbf{0}$ except the diagonal $x=y$;

7) $\langle \mathbf{L}_k, \mathbf{G}(\cdot,y)\rangle = 0,\ k=\overline{1,n}$.

4.3 Relation between Green's matrices

Let us write solutions to problems (3.5) with $\widetilde{g}_k = g_k = 0,\ k=\overline{1,n}$, via their Green's matrices

$$\mathbf{u}(x) = \int_0^1 \mathbf{G}^u(x,y)\mathbf{f}(y)\,dy, \qquad \mathbf{v}(x) = \int_0^1 \mathbf{G}^v(x,y)\mathbf{f}(y)\,dy,$$

respectively. Putting them into the formula (3.6), we can get

$$\int_0^1 \mathbf{G}^v(x,y)\mathbf{f}(y)\,dy = \int_0^1 \mathbf{G}^u(x,y)\mathbf{f}(y)\,dy - \sum_{k=1}^n \langle \mathbf{L}_k, \int_0^1 \mathbf{G}^u(\cdot,y)\mathbf{f}(y)\,dy\,\rangle \mathbf{v}^k(x)$$

or

$$\int_0^1 \mathbf{G}^v(x,y)\mathbf{f}(y)\,dy = \int_0^1 \big(\mathbf{G}^u(x,y) - \sum_{k=1}^n \mathbf{v}^k(x)\langle \mathbf{L}_k, \mathbf{G}^u(\cdot,y)\rangle\big)\mathbf{f}(y)\,dy.$$

From here we obtain the following formula.

Lemma 3. *The relation*

$$\mathbf{G}^v(x,y) = \mathbf{G}^u(x,y) - \sum_{k=1}^n \mathbf{v}^k(x)\langle \mathbf{L}_k, \mathbf{G}^u(\cdot,y)\rangle \tag{4.2}$$

between two Green's matrices for the problems (3.5) *is valid.*

4.4 Nonlocal boundary value problem

Let us investigate the unique solution $\mathbf{u}$ to the problem with nonlocal boundary conditions

$$\mathbf{u}' = \mathbf{A}\mathbf{u} + \mathbf{f}, \tag{4.3}$$

$$\langle \mathbf{L}_k, \mathbf{u}\rangle := \langle \mathbf{l}_k, \mathbf{u}\rangle - \gamma_k\langle \mathbf{n}_k, \mathbf{u}\rangle = g_k, \quad k=\overline{1,n}, \tag{4.4}$$

where $\mathbf{l}_k$ describes the classical part of conditions (4.4) and $\mathbf{n}_k$ describes the nonlocal part, $\gamma_k \in \mathbb{R},\ k=\overline{1,n}$. We suppose that the classical problem (4.3)–(4.4) ($\gamma_k = 0,\ k=\overline{1,n}$) also has the unique solution $\mathbf{u}^{\mathrm{cl}}$. Putting these solutions into the formula (3.6), we obtain the representation of the unique solution to the nonlocal boundary value problem (4.3)–(4.4) via the unique solution to the classical problem only as given bellow

$$\mathbf{u} = \mathbf{u}^{\mathrm{cl}} + \sum_{k=1}^n \gamma_k\langle \mathbf{n}_k, \mathbf{u}^{\mathrm{cl}}\rangle \mathbf{v}^k.$$

Here $\mathbf{v}^j$, $j = \overline{1,n}$, are the biorthogonal fundamental system of the nonlocal boundary value problem (4.3)–(4.4). Applying (4.2) to the Green's matrix $\mathbf{G}$ of the nonlocal boundary value problem and the Green's matrix $\mathbf{G}^{\mathrm{cl}}$ of the classical problem, we obtain very analogous relation.

Lemma 4. *The relation*

$$\mathbf{G}(x,y) = \mathbf{G}^{\mathrm{cl}}(x,y) + \sum_{k=1}^{n} \gamma_k \mathbf{v}^k(x) \langle \mathbf{n}_k, \mathbf{G}^{\mathrm{cl}}(\cdot, y)\rangle$$

between the Green's matrix and the Green's matrix for the classical problem is valid.

5 n-th order ordinary differential equations with nonlocal conditions

Now we are going to apply obtained results to the n-th order ordinary differential equation with nonlocal conditions

$$u^{(n)} + a_{n-1}(x)u^{(n-1)} + \ldots + a_1(x)u' + a_0(x)u = f(x), \; x \in [0,1], \tag{5.1}$$

$$\langle L_k, u\rangle := \sum_{i=1}^{n} \langle L_{ki}, u^{(i-1)}\rangle = g_k, \quad k = \overline{1,n}, \tag{5.2}$$

where $a_j, f \in C[0,1]$, $j = \overline{0, n-1}$, and $L_{ki} \in C^*[0,1]$, $g_k \in \mathbb{R}$, $i,k = \overline{1,n}$. First, introducing notation $u_j = u^{(j-1)}$, $j = \overline{1,n}$, we rewrite the problem (5.1)–(5.2) into the equivalent first order system for n equations as follows

$$\begin{aligned} u_j' &= u_{j+1}, \quad j = \overline{1, n-1}, \\ u_n' &= f - a_0 u_1 - a_1 u_2 - \ldots - a_{n-1} u_n \end{aligned} \tag{5.3}$$

with nonlocal conditions

$$\sum_{i=1}^{n} \langle L_{ki}, u_i\rangle = g_k, \quad k = \overline{1,n}.$$

Denoting the vector of unknown functions $\mathbf{u} = (u_1, u_2, \ldots, u_n)^\top$, this system can also be written in the vectorial form

$$\begin{aligned} &\mathbf{u}' = \mathbf{A}\mathbf{u} + \mathbf{f}, \\ &\langle \mathbf{L}_k, \mathbf{u}\rangle := \textstyle\sum_{i=1}^{n} \langle L_{ki}, u_i\rangle = g_k, \quad k = \overline{1,n} \end{aligned} \tag{5.4}$$

with the n-th order square matrix and the right hand side

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ & & & \ldots & & \\ 0 & 0 & 0 & \ldots & 0 & 1 \\ -a_0 & -a_1 & -a_2 & \ldots & -a_{n-2} & -a_{n-1} \end{pmatrix}, \qquad \mathbf{f} = \begin{pmatrix} 0 \\ 0 \\ \ldots \\ 0 \\ f \end{pmatrix}.$$

5.1 Green's function via Green's matrix

Since (5.1)–(5.2) and (5.4) describe the same problem but in different forms, the problem (5.1)–(5.2) has the unique solution $u \in C^n[0,1]$ if and only if the system (5.3) has the unique vectorial solution $\mathbf{u}$, where $u_i = u^{(i-1)} \in C[0,1]$, $i = \overline{1,n}$, and vice versa. Let us say $g_k = 0$, $k = \overline{1,n}$, and take their solutions, i.e. the unique solution to (5.1)–(5.2)

$$u(x) = \int_0^1 G(x,y) f(y)\, dy, \qquad x \in [0,1] \tag{5.5}$$

described by the Green's function $G(x,y)$ of the problem (5.1)–(5.2), and the unique solution to the system (5.4)

$$\mathbf{u}(x) = \int_0^1 \mathbf{G}(x,y) \mathbf{f}(y)\, dy, \qquad x \in [0,1] \tag{5.6}$$

represented by the Green's matrix $\mathbf{G}(x,y)$ of the system (5.4). Simplifying (5.6), we get

$$u_i(x) = \int_0^1 G_{in}(x,y) f(y)\, dy, \qquad i = \overline{1,n}.$$

Since $u_i = u^{(i-1)}$, $i = \overline{1,n}$, we differentiate (5.5) and obtain the following result.

Lemma 5. *The last column elements of Green's matrix for the problem* (5.4) *which corresponds to the scalar problem* (5.1)–(5.2) *are*

$$G_{in}(x,y) = \frac{\partial^{i-1}}{\partial x^{i-1}} G(x,y), \qquad i = \overline{1,n}. \tag{5.7}$$

Corollary 1. The Green's function for the problem (5.1)–(5.2) can be represented by the function from the Green's matrix of the system (5.4) by

$$G(x,y) = G_{1n}(x,y).$$

Remark 1. Thus, their properties (discontinuities, jumps, properties of derivatives) have to be the same. Indeed, according to Roman [13], on particular rectangles $x \in [0,1]$, $y_{l-1} < y < y_l$ for $l = \overline{1,N}$ (they depend on nonlocal conditions (5.2)), the Green's function $G(x,y)$ and its partial derivatives from the first to $(n-1)$-th order in x, i.e. $(\partial^j/\partial x^j)G(x,y)$, $j = \overline{0,n-1}$, are continuous and, additionally, the $(n-1)$-th order partial derivative has the jump on the diagonal

$$\frac{\partial^{n-1}}{\partial x^{n-1}} G(y+0,y) - \frac{\partial^{n-1}}{\partial x^{n-1}} G(y-0,y) = 1.$$

All these properties of the Green's function confirm the properties, given in subsection 4.2, for the elements G_{in} of the Green's matrix. Moreover, here we obtain additional smoothness properties $G_{in} \in C^{n-i}[0,1]$ except the diagonal $x = y$ and discontinuity points $y_1, y_2, \ldots$

5.2 Green's matrix via Green's function

Formulas (5.7) represent the last column of the Green's matrix $\mathbf{G}(x,y)$. To find the next to last column, we consider the system (5.4) with the different right hand side

$$\begin{aligned} &\mathbf{u}' = \mathbf{A}\mathbf{u} + \widetilde{\mathbf{f}}, \\ &\langle \mathbf{L}_k, \mathbf{u} \rangle = 0, \quad k = \overline{1,n}, \end{aligned} \tag{5.8}$$

where $\mathbf{u} = (u_1, u_2, \ldots, u_n)^\top$ and $\widetilde{\mathbf{f}} = (0, \ldots, 0, f_{n-1}, 0)^\top$. The solution to this system is represented only by the next to last column of the Green's matrix

$$u_i(x) = \int_0^1 G_{i,n-1}(x,y) f_{n-1}(y)\,dy, \qquad i = \overline{1,n}. \tag{5.9}$$

From the structure of equations (5.8), we get $u_i = u_1^{(i-1)}$, $i = \overline{1,n-1}$ and $u_n = u_1^{(n-1)} - f_{n-1}$. Now we observe that the system (5.8) is equivalent to the problem (5.1)–(5.2) for the function u_1 with $f = f'_{n-1} + a_{n-1} f_{n-1}$, if $f_{n-1} \in C^1[0,1]$, and $g_k = \langle L_{kn}, f_{n-1} \rangle$, $k = \overline{1,n}$. By [13], it has the solution

$$u_1(x) = \sum_{k=1}^n g_k v^k(x) + \int_0^1 G(x,y) f(y)\,dy.$$

As in subsection 5.1, we obtain the solution

$$\begin{aligned} u_i(x) = &\sum_{k=1}^n \langle L_{kn}, f_{n-1} \rangle (v^k(x))^{(i-1)} \\ &+ \int_0^1 \frac{\partial^{i-1} G(x,y)}{\partial x^{i-1}} \big(f'_{n-1}(y) + a_{n-1}(y) f_{n-1}(y)\big)\,dy \end{aligned} \tag{5.10}$$

for $i = \overline{1,n-1}$. For $u_n = u_1^{(n-1)} - f_{n-1}$, we obtain the analogous expression

$$\begin{aligned} u_n(x) = &\sum_{k=1}^n \langle L_{kn}, f_{n-1} \rangle (v^k(x))^{(n-1)} \\ &+ \int_0^1 \frac{\partial^{n-1} G(x,y)}{\partial x^{n-1}} \big(f'_{n-1}(y) + a_{n-1}(y) f_{n-1}(y)\big)\,dy - f_{n-1}(x). \end{aligned} \tag{5.11}$$

Now we apply the integration by parts formula to the integral in formulae (5.10)–(5.11) for $i = \overline{1,n}$:

$$\begin{aligned} \int_0^1 \frac{\partial^{i-1} G(x,y)}{\partial x^{i-1}} f'_{n-1}(y)\,dy &= \sum_{l=1}^N \int_{y_{l-1}}^{y_l} \frac{\partial^{i-1} G(x,y)}{\partial x^{i-1}} f'_{n-1}(y)\,dy \\ = \sum_{l=1}^N \Big(\frac{\partial^{i-1} G(x,y)}{\partial x^{i-1}} f_{n-1}(y) \big|_{y=y_{l-1}+0}^{y=y_l-0} &- \int_{y_{l-1}}^{y_l} \frac{\partial^i G(x,y)}{\partial x^{i-1} \partial y} f_{n-1}(y)\,dy \Big) \\ = \sum_{l=1}^N \int_{y_{l-1}}^{y_l} \Big(\big(\delta(y - y_l) - \delta(y - y_{l-1})\big) &\frac{\partial^{i-1} G(x,y)}{\partial x^{i-1}} - \frac{\partial^i G(x,y)}{\partial x^{i-1} \partial y} \Big) f_{n-1}(y)\,dy. \end{aligned}$$

Remark 2. We remember that every considering condition (5.8) is of the form (1.3) or, applying the popular notation $\int_0^1 u_i(y)\delta(y-x)\,dy = u_i(x)$ for every component u_i with the *delta* function, conditions (5.8) can be rewritten by

$$\int_0^1 \mathbf{L}_k(y)\mathbf{u}(y)\,dy = 0 \tag{5.12}$$

for some row-matrix function $\mathbf{L}_k = (L_{k1}, L_{k2}, \dots, L_{kn})$ on $[0,1]$, where components may be described by the *delta* function as well. Moreover, we will use notation $\delta_x(y) := \delta(y-x)$. From (5.12), every functional $L_{kn} \in C^*[0,1]$ is also represented by an integral

$$\langle L_{kn}, f_{n-1}\rangle = \int_0^1 L_{kn}(y) f_{n-1}(y)\,dy, \quad k = \overline{1,n}.$$

We put obtained integrals into (5.10)–(5.11) and get the representation of the solutions (5.9) with the kernel, which is of the form

$$\begin{aligned} G_{i,n-1}(x,y) =& \frac{\partial^{i-1}}{\partial x^{i-1}} G(x,y) a_{n-1}(y) - \frac{\partial^i}{\partial x^{i-1}\partial y} G(x,y) \\ &+ \sum_{k=1}^n L_{kn}(y)(v^k(x))^{(i-1)} - H_{i-n+1}\delta_x(y), \qquad i = \overline{1,n} \end{aligned}$$

except the discontinuity points $y = y_l,\ l = \overline{0,N}$. Here we used the discrete Heaviside function $H_i = 1$ if $i > 0$ or $H_i = 0$ if $i \le 0$. The jumps of elements $G_{i,n-1}$ of the Green's matrix on $y = y_1,\ y_2,\ y_3, \dots$ can always be found from this formula as well.

Similarly, we find the m-th column of the Green's matrix investigating the problem (5.8) with $\widetilde{\mathbf{f}} = (0,\dots,0,f_m,0,\dots,0)^\top,\ m = \overline{1,n-1}$. Now we get $u_i = u_1^{(i-1)},\ i = \overline{1,m}$, and $u_{m+i} = u_1^{(m+i-1)} - f_m^{(i-1)},\ i = \overline{1,n-m}$. Now the system is equivalent to the problem (5.1)–(5.2) for the function u_1 with $f = \sum_{i=0}^{n-m} a_{m+i} f_m^{(i)}$, $a_n \equiv 1$, if $f_m \in C^{n-m}[0,1]$, and $g_k = \sum_{i=1}^{n-m} \langle L_{k,m+i}, f_m^{(i-1)}\rangle$, $k = \overline{1,n}$. According to [13], it has the solution

$$u_1(x) = \sum_{k=1}^n g_k v^k(x) + \int_0^1 G(x,y) f(y)\,dy,$$

which allows us to find the solution to the system, i.e.

$$\begin{aligned} u_i(x) =& \sum_{k=1}^n g_k (v^k(x))^{(i-1)} + \int_0^1 \frac{\partial^{i-1} G(x,y)}{\partial x^{i-1}} f(y)\,dy, \quad i = \overline{1,m}, \\ u_{m+i}(x) =& \sum_{k=1}^n g_k (v^k(x))^{(m+i-1)} + \int_0^1 \frac{\partial^{m+i-1} G(x,y)}{\partial x^{m+i-1}} f(y)\,dy - f_m^{(i-1)}(x), \end{aligned}$$

if $i = \overline{1,n-m}$. From (5.12) follows that every considering functional $L_{ki} \in C^*[0,1]$ can also be represented by $\langle L_{ki}, u\rangle = \int_0^1 L_{ki}(x) u(x)\,dx$. Moreover, we take functions $L_{ki} \in C^{i-2}[0,1]$, $i = \overline{2,n}$ for regular functionals, and rewrite the solution in the form

$$u_i(x) = \int_0^1 G_{im}(x,y) f_m(y)\,dy, \qquad i = \overline{1,n},$$

where

$$\begin{aligned} G_{im}(x,y) = &\sum_{j=0}^{n-m} (-1)^j \frac{\partial^j}{\partial y^j}\Bigl(a_{m+j}(y)\frac{\partial^{i-1}}{\partial x^{i-1}} G(x,y)\Bigr) \\ &+ \sum_{j=1}^{n-m}\sum_{k=1}^{n} (-1)^{j-1} L_{k,m+j}^{(j-1)}(y)(v^k(x))^{(i-1)} - H_{i-m}\delta_x^{(i-1-m)}(y) \end{aligned}$$

for $i = \overline{1,n}$ if $y \neq y_l$, $l = \overline{0,N}$. Here we used functionals

$$\langle \delta_x^{(i)}, u\rangle = \langle \delta_x^{(i)}(y), u(y)\rangle := (-1)^i u^{(i)}(x), \quad u \in C^i[0,1], \ \ i = \overline{0,n},$$

respectively.

Remark 3. We note that the representation of the Green's matrix is also valid if $\langle L_{ki}, u\rangle := \langle \delta_\xi, u\rangle = u(\xi)$ for every fixed $\xi \in [0,1]$. Precisely, we can put $L_{ki}(y) = \delta(y-\xi)$ into the last formula. Applying properties of generalized functions, we extended the last representation of the Green' function with the delta function, representing the one point condition, or every combination (1.3).

5.3 The second order problem

For example, let us consider the second order problem ($n = 2$)

$$\begin{aligned} &u_1' = u_2 + f_1, \quad u_2' = -a_0 u_1 - a_1 u_2 + f_2, \\ &\langle \mathbf{L}_k, \mathbf{u}\rangle := \langle L_{k1}, u_1\rangle + \langle L_{k2}, u_2\rangle = 0, \quad k = 1,2, \end{aligned}$$

where $f_1 \in C^1[0,1]$, $f_2 \in C[0,1]$. It has the Green's matrix $\mathbf{G}(x,y)$, which is of the form

$$\begin{pmatrix} a_1(y)G - G_y' + L_{12}(y)v^1(x) + L_{22}(y)v^2(x) & G \\ a_1(y)G_x' - G_{yx}'' + L_{12}(y)(v^1)'(x) + L_{22}(y)(v^2)'(x) - \delta_x(y) & G_x' \end{pmatrix}$$

if $y \neq y_l$, $l = \overline{0,N}$. Here we used the Green's function $G = G(x,y)$ for the problem

$$\begin{aligned} &u'' + a_1(x)u' + a_0(x)u = f(x), \quad x \in [0,1], \\ &\langle L_k, u\rangle := \langle L_{k1}, u\rangle + \langle L_{k2}, u'\rangle = 0, \quad k = 1,2. \end{aligned}$$

Example 1. Let us take a differential problem with the Bitsadze–Samarskii condition

$$\begin{aligned} &-u'' = f(x), \quad x \in [0,1], \\ &u(0) = 0, \quad u(1) = \gamma u(\xi), \end{aligned} \tag{5.13}$$

where $f \in C[0,1]$ is a real function and $\gamma \in \mathbb{R}$, $\xi \in (0,1)$. According to [13], it has the unique solution and the Green's function

$$G(x,y) = \begin{cases} y(1-x), & y \le x, \\ x(1-y), & x < y, \end{cases} + \frac{\gamma x}{1-\gamma\xi} \begin{cases} y(1-\xi), & y \le \xi, \\ \xi(1-y), & \xi < y, \end{cases} \tag{5.14}$$

if and only if $\gamma\xi \neq 1$. Denoting $u_1 = u$ and $u_2 = u'$, we rewrite the problem (5.13) into the equivalent system (5.14), i.e.

$$\begin{aligned} &u_1' = u_2 + f_1, \quad u_2' = f_2, \\ &u_1(0) = 0, \quad u_1(1) = \gamma u_1(\xi) \end{aligned}$$

with $f_1 = 0,\ f_2 = -f$. The previous system has the Green's matrix, which is of the form

$$\mathbf{G}(x,y) = \begin{pmatrix} -G_y'(x,y) & G(x,y) \\ -G_{yx}''(x,y) - \delta_x(y) & G_x'(x,y) \end{pmatrix}$$

if $y \neq 0,\ \xi,\ 1$. Applying (5.14), we can directly obtain the properties of the Green's function given in subsection 4.2.

Example 2. Let us now consider another differential problem with one integral condition

$$\begin{aligned} &-u'' = f(x), \quad x \in [0,1], \\ &u(0) = 0, \quad u'(1) = \gamma \int_0^1 u(x)\,dx, \end{aligned} \tag{5.15}$$

where $f \in C[0,1]$ is a real function and $\gamma \in \mathbb{R}$. From [13], we find the Green's function

$$G(x,y) = \begin{cases} y, & y \le x, \\ x, & x < y, \end{cases} + \frac{\gamma}{2-\gamma} \cdot xy(2-y).$$

It exists if and only if $\gamma \neq 2$. As in Example 1, we obtain the system

$$u_1' = u_2 + f_1, \quad u_2' = f_2, \qquad u_1(0) = 0, \quad u_2(1) = \gamma \int_0^1 u_1(x)\,dx,$$

which is coincident with the problem (5.15) if $f_1 = 0,\ f_2 = -f$ and $u_1 = u,\ u_2 = u'$. Thus, the Green's matrix to the last system is given by

$$\mathbf{G}(x,y) = \begin{pmatrix} -G_y'(x,y) & G(x,y) \\ -G_{yx}''(x,y) - \delta_x(y) & G_x'(x,y) \end{pmatrix}$$

if $y \neq 0,\ 1$. Here we used $L_{12}(y) = 0$ and $L_{22}(y) = \delta(y-1)$ since $\langle L_{12}, u_2\rangle \equiv 0$ and $\langle L_{22}, u_2\rangle = u_2(1)$.

5.4 Final remarks

We remark that the investigation of the Green's matrix is still unfinished. As other authors [1,3], we analyzed the properties of the Green's matrix except the discontinuity points $y_i,\ i = \overline{0,N}$, only. In the last section, the representation of the Green's matrix via the Green's function is derived except the discontinuity points as well. Analyzing the behaviour and properties of the Green's matrix around and on the discontinuity points, we ran into fully unruled but interesting questions: Does the obtained Green's matrix restore the solution to the system if we ignore the discontinuity points? How to restore the solution via the investigated Green's matrix if the discontinuity point plays an important role representing the solution? We still continue the investigation and plan to present obtained results about discontinuities in the separate our work.

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