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Difference Equations in a Multidimensional Space*

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Abstract. One considers a general difference equation in a multidimensional space with continuous coefficients in the space of integrable functions. Necessary and sufficient conditions for a Fredholm property are obtained with a help of the Fourier transform and the Riemann boundary value problem. For simplest cases solvability conditions and formula of a general solution for the difference equation are given.

Keywords: difference equation, symbol, singular integral equation, Riemann boundary value problem, index.

AMS Subject Classification: 39A06; 35S05.

1 Introduction

We consider a general difference equation in a multidimensional space of the following type

$$\sum_{|k|=0}^{+\infty} a_k(x)u(x+\beta_k) = v(x), \quad x \in D, \tag{1.1}$$

where D is $\mathbb{R}^m$ or $\mathbb{R}^m_+ = \{x \in \mathbb{R}^m : \ x = (x_1, \cdots, x_m), x_m > 0\}$, k is multi-index $k = (k_1, \cdots, k_m)$, $\beta_k = (\beta_{k_1}, \cdots, \beta_{k_m}) \in D$. Our main goal is to describe a Fredholm property for such equations. Such equations have variable coefficients, and only for one-dimensional equations on a straight line with constant coefficients one can construct exact solution [4, 6, 11]. We'll use methods of the theory of singular integral equations [2, 3, 5, 7], boundary value problems [1], and the Wiener–Hopf technique [8]. Some results related to certain classes of difference equations are described in papers [12, 13, 15, 16, 17].

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This paper is devoted to equations with a continual variable. The case of a discrete variable will be considered in a separate paper because there are certain principal distinctions.

2 The Fourier transform and symbol

If we consider the equation with constant coefficients in the whole space $\mathbb{R}^m$

$$\sum_{|k|=0}^{+\infty} a_k u(x+\beta_k) = v(x), \quad x \in \mathbb{R}^m, \tag{2.1}$$

then we can use the Fourier transform

$$\tilde{u}(\xi) = \int_{\mathbb{R}^m} e^{ix\cdot\xi} u(x)dx$$

and obtain an equivalent equation in the space $L_2(\mathbb{R}^m)$

$$\sigma(\xi)\tilde{u}(\xi) = \tilde{v}(\xi),$$

where

$$\sigma(\xi) = \sum_{k=0}^{+\infty} a_k e^{i\beta_k\cdot\xi}, \quad \xi \in \mathbb{R}^m. \tag{2.2}$$

It implies necessary and sufficient condition for a unique solvability of the equation (2.1): if $\sigma \in L_\infty(\mathbb{R}^m)$ then this is

$$\operatorname*{ess\,inf}_{\xi\in\mathbb{R}^m} |\sigma(\xi)| > 0.$$

Unfortunately we can't use this approach if we are in the space $\mathbb{R}^m_+$ and consider the equation (1.1) with constant coefficients

$$\sum_{|k|=0}^{+\infty} a_k u(x+\beta_k) = v(x), \quad x \in \mathbb{R}^m_+, \tag{2.3}$$

because we have no description for Fourier image of the space $L_2(\mathbb{R}^m_+)$. Hence the first step is to obtain such description.

This description is a very simple and it is described in [1]. The book [1] is devoted to constructing theory of pseudo differential equations on manifolds with a smooth boundary but methods introduced in the book are applicable to our situation also. Here we'll give a brief sketch of main ideas from [1].

If $u \in L_2(\mathbb{R}^m)$ then

$$FP_\pm u = \Pi_\pm Fu,$$

where $P_\pm$ is the projection operator on $\mathbb{R}^m_\pm$, i.e. $(P_\pm u)(x) = u(x)$ if $x \in \mathbb{R}^m_\pm$, and $(P_\pm u)(x) = 0$ otherwise, $\Pi_\pm$ is the following operator

$$(\Pi_\pm u)(\xi) = \frac{i}{2\pi}_{\tau\to 0+} \int_{-\infty}^{+\infty} \frac{u(\xi', \eta_m)d\eta_m}{\xi_m \pm i\tau - \eta_m}, \quad \xi' = (\xi_1, \cdots, \xi_{m-1}).$$

This relation is a very important for constructing a solution of the difference equation (2.3).

Let's introduce the following notations. Let $\mathcal{A}, \mathcal{B}$ be difference operators of the type

$$(\mathcal{A}u)(x) = \sum_{|k|=0}^{+\infty} a_k u(x+\alpha_k), \quad (\mathcal{B}u)(x) = \sum_{|k|=0}^{+\infty} b_k u(x+\beta_k), \quad x \in \mathbb{R}^m.$$

The functions

$$\sigma_{\mathcal{A}}(\xi) = \sum_{k=0}^{+\infty} e^{i\alpha_k \cdot \xi}, \quad \sigma_{\mathcal{B}}(\xi) = \sum_{k=0}^{+\infty} e^{i\beta_k \cdot \xi}, \quad \xi \in \mathbb{R}^m$$

are called the symbols of the operators $\mathcal{A}, \mathcal{B}$ respectively.

Obviously these operators $\mathcal{A}, \mathcal{B}$ are linear bounded operators $L_2(\mathbb{R}^m) \longrightarrow L_2(\mathbb{R}^m)$ if $\sigma_{\mathcal{A}}, \sigma_{\mathcal{B}} \in L_\infty(\mathbb{R}^m)$.

The equation (2.3) can be written in an operator form as follows

$$(P_+\mathcal{A}u)(x) = v(x), \quad x \in \mathbb{R}^m_+. \tag{2.4}$$

One can consider also more general equation in the space $L_2(\mathbb{R}^m)$

$$(\mathcal{A}P_+ + \mathcal{B}P_-)U = V \tag{2.5}$$

and conclude that the equation (2.4) is equivalent to the equation (2.5) with $\mathcal{B} \equiv I$, where I is identity operator. This equivalence means that if we know the solution of (2.4) we can write the solution of

$$(\mathcal{A}P_+ + IP_-)U = V$$

and vice versa. That's why we'll consider more general equation (2.5).

After applying the Fourier transform to (2.5) we obtain

$$\sigma_{\mathcal{A}}(\xi)(\mathit{\Pi}_+\tilde{U})(\xi) + \sigma_{\mathcal{B}}(\xi)(\mathit{\Pi}_-\tilde{U})(\xi) = \tilde{V}(\xi) \tag{2.6}$$

and the last equation (2.6) is one-dimensional characteristic singular integral equation with the parameter ξ'. Such conclusion we have because the introduced operators $\mathit{\Pi}_\pm$ are connected to the operator (Plemelj–Sokhotskii formulas)

$$(Hu)(\xi) = \frac{1}{\pi i} v.p. \int_{-\infty}^{+\infty} \frac{u(\xi', \eta_m)d\eta_m}{\xi_m - \eta_m},$$

which is well-known as the Hilbert transform or one-dimensional singular integral operator [2, 3, 5, 7].

3 Singular integral equation and Riemann boundary value problem

The equation (2.6) can be rewritten as the Riemann boundary value problem

$$\begin{aligned}(\mathit{\Pi}_+\tilde{U})(\xi', \xi_m) = &-\sigma_{\mathcal{A}}^{-1}(\xi', \xi_m)\sigma_{\mathcal{B}}(\xi', \xi_m)(\mathit{\Pi}_-\tilde{U})(\xi', \xi_m) \\ &+\sigma_{\mathcal{A}}^{-1}(\xi', \xi_m)\tilde{V}(\xi', \xi_m), \quad \xi_m \in \mathbb{R},\end{aligned}$$

or as the one-dimensional singular integral equation

$$\frac{\sigma_{\mathcal{A}}(\xi',\xi_m)+\sigma_{\mathcal{B}}(\xi',\xi_m)}{2}\tilde{U}(\xi',\xi_m)+\frac{\sigma_{\mathcal{A}}(\xi',\xi_m)-\sigma_{\mathcal{B}}(\xi',\xi_m)}{2}$$
$$\times\,(H\tilde{U})(\xi',\xi_m)=\tilde{V}(\xi',\xi_m),\quad \xi_m\in\mathbb{R}$$

with a parameter $\xi' \in \mathbb{R}^{m-1}$.

The theory for such equations or equivalently Riemann boundary value problems is well-known [2, 3, 5, 7]. The index of such Riemann boundary value problem plays key role in describing construction for a solution. For our case we have only one difficulty related to a parameter ξ'. We'll give definition for the index in connection with our case.

Let's denote

$$\sigma(\xi',\xi_m)\equiv-\sigma_{\mathcal{A}}^{-1}(\xi',\xi_m)\sigma_{\mathcal{B}}(\xi',\xi_m)$$

and suppose that $\sigma \in C(\dot{\mathbb{R}}^m), \dot{\mathbb{R}}^m$ is a compactification of $\mathbb{R}^m$.

Definition 1. Symbol $\sigma(\xi)$ is called elliptic if $\sigma(\xi)\neq 0$, $\forall \xi \in \dot{\mathbb{R}}^m$.

Fix $\xi' \in \mathbb{R}^{m-1}$ and define

$$æ(\xi')\equiv Ind\ \sigma=\frac{1}{2\pi}\int_{-\infty}^{+\infty} d\arg\sigma(\xi',\xi_m).$$

Remark 1. This index is an integer, and indeed it doesn't depend on ξ' if $m \geq 2$ (homotopy property). The case $m = 1$ is a very specific one (see [14]). So we have $æ(\xi') = æ$.

4 Solvability condition

Definition 2. Factorization of the elliptic symbol $\sigma(\xi)$ is called its representation in the form

$$\sigma(\xi)=\sigma_+(\xi)\cdot\sigma_-(\xi),$$

where the factors $\sigma_\pm$ admit an analytical continuation into complex half-planes $\mathbb{C}_\pm$ and $\sigma_\pm \in L_\infty(\mathbb{R})$.

Such factorization (in some more general sense) exists for all cases and can be constructed by the Hilbert transform H [2, 3, 5, 7].

Now we are ready to formulate a basic result on unique solvability of the equation (2.4).

Theorem 1. *Let $\sigma(\xi)$ be an elliptic symbol. Then for unique solvability of the equation* (2.5) *in the space $L_2(\mathbb{R}^m)$ it is necessary and sufficient* $æ = 0$.

Proof. We see that our operator (and equation) are one-dimensional one, hence we can use one-dimensional theory (see [14]). According to the needed result the equation (2.4) has a unique solution in $L_2(\mathbb{R}^m_+)$ if the condition $æ = 0$ holds. □

Of course there are another cases when the index is not zero. One-dimensional constructions for such situations are described in [14]. To adapt such constructions to multi-dimensional case we need more large scale of functional spaces than $L_2(\mathbb{R}^m_+)$.

5 Mapping properties of difference operators

DEFINITION 3. The space $H^s(\mathbb{R}^m), s \in \mathbb{R}$, consists of (generalized) functions for which the following norm

$$\|u\|_s = \left(\int_{\mathbb{R}^m} |\tilde{u}(\xi)|^2(1+|\xi|)^{2s}d\xi\right)^{1/2}$$

is a finite number.

This space is a Hilbert space, and the Schwartz space $S(\mathbb{R}^m)$ consisting of infinitely differentiable rapidly decreasing at infinity functions is dense in $H^s(\mathbb{R}^m)$ [1]. Obviously $H^0(\mathbb{R}^m) = L_2(\mathbb{R}^m)$.

Lemma 1. *Let $\sigma_{\mathcal{A}} \in L_\infty(\mathbb{R}^m)$. Then the operator $\mathcal{A}$ is a linear bounded operator $H^s(\mathbb{R}^m) \longrightarrow H^s(\mathbb{R}^m)$.*

Proof. Indeed, for $u \in S(\mathbb{R}^m)$

$$\begin{aligned}(F\mathcal{A}u)(\xi) &= \int_{\mathbb{R}^m} e^{ix\cdot\xi}\Big(\sum_{|k|=0}^{\infty} a_k u(x+\alpha_k)\Big)dx \\ &= \sum_{|k|=0}^{\infty} a_k \int_{\mathbb{R}^m} e^{ix\cdot\xi}u(x+\alpha_k)dx = \sum_{|k|=0}^{\infty} a_k e^{i\alpha_k\cdot\xi}\tilde{u}(\xi) = \sigma_{\mathcal{A}}(\xi)\cdot\tilde{u}(\xi),\end{aligned}$$

under the condition that sum and integral can be re-arranged, it's possible according to our assumptions, and further

$$\begin{aligned}\|\mathcal{A}u\|_s^2 &= \int_{\mathbb{R}^m} |\sigma_{\mathcal{A}}(\xi)\cdot\tilde{u}(\xi)|^2(1+|\xi|)^{2s}d\xi \\ &\leq \Big(\operatorname*{ess\,sup}_{\xi\in\mathbb{R}^m}|\sigma_{\mathcal{A}}(\xi)|\Big)^2 \int_{\mathbb{R}^m} |\tilde{u}(\xi)|^2(1+|\xi|)^{2s}d\xi = \|\sigma_{\mathcal{A}}\|^2_{L_\infty}\|u\|_s^2.\end{aligned}$$

□

Remark 2. It seems that the operator $\mathcal{A}$ is a pseudo differential operator with the symbol $\sigma_{\mathcal{A}}(\xi)$. It is so but the formula

$$F^{-1}_{\xi\to x}\left(\sigma_{\mathcal{A}}(\xi)\cdot\tilde{u}(\xi)\right)$$

will define a very specific integral operator of convolution type with a non-regular kernel

$$K_{\mathcal{A}}(x) = \sum_{|k|=0}^{\infty} p_k\delta(x+\alpha_k),$$

so that

$$F^{-1}_{\xi\to x}\left(\sigma_{\mathcal{A}}(\xi)\cdot\tilde{u}(\xi)\right)=\int_{\mathbb{R}^m}\Big(\sum_{|k|=0}^{\infty}p_k\delta(x+\alpha_k-y)\Big)u(y)\,dy.$$

6 Non-vanishing indices

Here we'll apply our one-dimensional results from [14] to obtain solvability picture for a multi-dimensional case.

6.1 Positive index

Let $æ\in\mathbb{N}$. First we introduce a function

$$\omega(\xi',\xi_m)=\left(\frac{\xi_m-i|\xi'|}{\xi_m+i|\xi'|}\right)^{æ},$$

which belongs to $L_\infty(\mathbb{R}^m)$.

Evidently the functions $z\pm i|\xi'|$ for fixed $\xi'\in\mathbb{R}^{m-1}$ are analytical functions in complex half-planes $\mathbb{C}_\pm$. Moreover

$$Ind\,\frac{\xi_m-i|\xi'|}{\xi_m+i|\xi'|}\equiv\frac{1}{2\pi}\int_{-\infty}^{+\infty}d\arg\frac{\xi_m-i|\xi'|}{\xi_m+i|\xi'|}=1.$$

According to the index property a function $\omega^{-1}(\xi',\xi_m)\sigma(\xi',\xi_m)$ has a vanishing index, and it can be factorized in a sense of the definition 1:

$$\omega^{-1}(\xi',\xi_m)\sigma(\xi',\xi_m)=\sigma_+(\xi',\xi_m)\sigma_-(\xi',\xi_m),$$

so we have $\sigma(\xi',\xi_m)=\omega(\xi',\xi_m)\sigma_+(\xi',\xi_m)\sigma_-(\xi',\xi_m)$, where

$$\sigma_\pm(\xi',\xi_m)=\exp(\Psi^\pm(\xi',\xi_m)),\quad \Psi^\pm(\xi',\xi_m)=\frac{1}{2\pi i}\lim_{\tau\to 0+}\int\limits_{-\infty}^{+\infty}\frac{\ln(\omega^{-1}\sigma)(\xi,\eta_m)d\eta_m}{\xi_m\pm i\tau-\eta_m}.$$

Further we'll apply the Wiener–Hopf technique to the equation (2.6) and write it in a convenient form:

$$\sigma_{\mathcal{A}}(\xi)\tilde{U}_+(\xi)+\sigma_{\mathcal{B}}(\xi)\tilde{U}_-(\xi)=\tilde{V}(\xi).$$

Taking into account our above calculations we have

$$\tilde{U}_+(\xi)+\omega(\xi',\xi_m)\sigma_+(\xi',\xi_m)\sigma_-(\xi',\xi_m)\tilde{U}_-(\xi)=\sigma^{-1}_{\mathcal{A}}(\xi',\xi_m)\tilde{V}(\xi),$$

or in other words

$$\sigma^{-1}_+(\xi',\xi_m)\tilde{U}_+(\xi)+\omega(\xi',\xi_m)\sigma_-(\xi',\xi_m)\tilde{U}_-(\xi)=\sigma^{-1}_+(\xi',\xi_m)\sigma^{-1}_{\mathcal{A}}(\xi',\xi_m)\tilde{V}(\xi).$$

Let's introduce the following notations.We denote for shortness

$$\sigma^{-1}_+(\xi',\xi_m)\sigma^{-1}_{\mathcal{A}}(\xi',\xi_m)\tilde{V}(\xi)\equiv\tilde{h}(\xi',\xi_m)$$

and define the spaces $A(\mathbb{R}^m), B(\mathbb{R}^m)$ as subspaces of functions from $L_2(\mathbb{R}^m)$ which admit an analytical continuation on the last variable ξ_m under almost all fixed $\xi' \in \mathbb{R}^{m-1}$ into complex half-planes $\mathbb{C}_+, \mathbb{C}_-$ respectively so that

$$A(\mathbb{R}^m) \oplus B(\mathbb{R}^m) = L_2(\mathbb{R}^m) \quad for \quad a.a.\ \xi' \in \mathbb{R}^{m-1}.$$

Since $\tilde{h} \in L_2(\mathbb{R}^m)$ it can be represented as a sum $\tilde{h} = \tilde{h}_+ + \tilde{h}_-$, where $\tilde{h}_+ \in A(\mathbb{R}^m), \tilde{h}_- \in B(\mathbb{R}^m)$, and $\tilde{h}_\pm = \Pi_\pm \tilde{h}$. Thus we have

$$\sigma_+^{-1}(\xi', \xi_m)\tilde{U}_+(\xi) + \omega(\xi', \xi_m)\sigma_-(\xi', \xi_m)\tilde{U}_-(\xi) = \tilde{h}_+ + \tilde{h}_-$$

and taking into account a form of the function $\omega(\xi', \xi_m)$ we write

$$\begin{aligned}(\xi_m + i|\xi'|)^{æ}\sigma_+^{-1}(\xi', \xi_m)\tilde{U}_+(\xi', \xi_m) + (\xi_m - i|\xi'|)^{æ}\sigma_-(\xi', \xi_m)\tilde{U}_-(\xi', \xi_m)\\ = (\xi_m + i|\xi'|)^{æ}\tilde{h}_+(\xi', \xi_m) + (\xi_m + i|\xi'|)^{æ}\tilde{h}_-(\xi', \xi_m)\end{aligned}$$

and hence

$$\begin{aligned}(\xi_m + i|\xi'|)^{æ}\sigma_+^{-1}(\xi', \xi_m)\tilde{U}_+(\xi', \xi_m) - (\xi_m + i|\xi'|)^{æ}\tilde{h}_+(\xi', \xi_m)\\ = (\xi_m + i|\xi'|)^{æ}\tilde{h}_-(\xi', \xi_m) - (\xi_m - i|\xi'|)^{æ}\sigma_-(\xi', \xi_m)\tilde{U}_-(\xi', \xi_m). \quad (6.1)\end{aligned}$$

The left hand side of the equation (6.1) belongs to $A(\mathbb{R}^m)$ and its analytical continuation into upper complex half-plane on the variable ξ_m has a pole of order æ at infinity, and the right hand side belongs to the space $B(\mathbb{R}^m)$ and has the same pole. Then taking into account the generalized Liouville theorem [2,7] we conclude that both left had side and right hand side is a polynomial of order no more than æ -1. Since we are interested in left hand side we write

$$(\xi_m + i|\xi'|)^{æ}\sigma_+^{-1}(\xi', \xi_m)\tilde{U}_+(\xi', \xi_m) - (\xi_m + i|\xi'|)^{æ}\tilde{h}_+(\xi', \xi_m) = \sum_{k=0}^{æ-1} c_k(\xi')\xi_m^k,$$

where $c_k, k = 0, \cdots, æ - 1$ are arbitrary functions depending on a parameter ξ'. So we have the following formula

$$\tilde{U}_+(\xi', \xi_m) = \sigma_+(\xi', \xi_m)\tilde{h}_+(\xi', \xi_m) + (\xi_m + i|\xi'|)^{-æ}\sigma_+(\xi', \xi_m)\sum_{k=0}^{æ-1} c_k(\xi')\xi_m^k.$$

Obviously the first summand of the last formula belongs to the $L_2(\mathbb{R}^m_+)$. Since $\sigma_+ \in L_\infty(\mathbb{R}^m)$ then summands $(\xi_m + i|\xi'|)^{-æ}c_k(\xi')\xi_m^k$, should belong to the $L_2(\mathbb{R}^m_+)$, $k = 0, \cdots, æ - 1$. It means that

$$\int_{\mathbb{R}^m} |\xi_m + i|\xi'||^{-2æ}|c_k(\xi')|^2|\xi_m|^{2k}d\xi < +\infty. \quad (6.2)$$

Passing to a repeated integral we first estimate a one-dimensional integral

$$\int_{-\infty}^{+\infty} \left|\xi_m + i|\xi'|\right|^{-2æ}|\xi_m|^{2k}d\xi_m.$$

It exists at infinity only if $-2æ + 2k < -1$, i.e. $k < æ - 1/2$.

More precise estimate gives the following for large $|\xi'|$

$$\int_{-\infty}^{+\infty} |\xi_m + i|\xi'||^{-2æ}|\xi_m|^{2k} d\xi_m \sim \int_0^{+\infty} |\xi|^{-2æ}|\xi_m|^{2k} d\xi_m \sim (1+|\xi'|)^{-2æ+2k+1}.$$

Returning to the integral (6.2) we see that it exists only if

$$\int_{\mathbb{R}^{m-1}} |c_k(\xi')|^2(1 + |\xi'|)^{-2æ+2k+1} d\xi' < +\infty.$$

The last means that $c_k \in H^{s_k}(\mathbb{R}^{m-1})$, $s_k = -æ + k + 1/2$, $k = 0, \cdots, æ - 1$.

Thus we have proved the following theorem.

Theorem 2. *Let $\sigma(\xi)$ be an elliptic symbol, æ is its index of factorization with factors $\sigma_\pm(\xi)$. If $æ \in \mathbb{N}$ then the equation* (2.5) *has many solutions in the space $L_2(\mathbb{R}^m)$, and formula for a general solution in Fourier image*

$$\tilde{U}_+(\xi', \xi_m) = \sigma_+(\xi', \xi_m)\tilde{h}_+(\xi', \xi_m) + (\xi_m + i|\xi'|)^{-æ}\sigma_+(\xi', \xi_m) \sum_{k=0}^{æ-1} c_k(\xi')\xi_m^k$$

holds, where lv is an arbitrary continuation v on $\mathbb{R}^m$, $c_k \in H^{s_k}(\mathbb{R}^{m-1})$, $s_k = -æ + k + 1/2$, $k = 0, \cdots, æ - 1$, are arbitrary functions.

Corollary 1. If under assumptions of the theorem 3 $v \equiv 0$ then a general solution of the equation

$$(\mathcal{A}u)(x) = 0, \quad x \in \mathbb{R}^m_+$$

has the form

$$\tilde{u}(\xi) = (\xi_m + i|\xi'|)^{-æ}\sigma_+(\xi', \xi_m) \sum_{k=0}^{æ-1} c_k(\xi')\xi_m^k.$$

Note. If we are interested in the equation (2.4) we need to put $\sigma(\xi) = \sigma_{\mathcal{A}}^{-1}(\xi)$ to obtain this assertion.

6.2 Negative index

Theorem 3. *Let $\sigma(\xi)$ be an elliptic symbol and $æ < 0$. The equation* (2.5) *has a solution $U \in L_2(\mathbb{R}^m)$ iff the right hand side V satisfies the following conditions:*

$$\int_{-\infty}^{+\infty} \frac{\sigma_+^{-1}(\xi', \eta_m)\sigma_{\mathcal{A}}^{-1}(\xi', \eta_m)\tilde{V}(\xi', \eta_m)d\eta_m}{(\eta_m + i|\xi'|)^{k+1}} = 0, \quad k = 0, 1, \cdots, |æ|.$$

Proof. Initial point for the proof is the quality (6.1). Then we conclude that both the left hand side and the right hand side are zero because we have a zero of order $-æ$ at infinity. Hence

$$\begin{aligned}(\xi_m + i|\xi'|)^{æ}\sigma_+^{-1}(\xi', \xi_m)\tilde{U}_+(\xi', \xi_m) - (\xi_m + i|\xi'|)^{æ}\tilde{h}_+(\xi', \xi_m) \\ = (\xi_m + i|\xi'|)^{æ}\tilde{h}_-(\xi', \xi_m) - (\xi_m - i|\xi'|)^{æ}\sigma_-(\xi', \xi_m)\tilde{U}_-(\xi', \xi_m) = 0\end{aligned}$$

and we have

$$\tilde{U}_+(\xi',\xi_m)=\sigma_+(\xi',\xi_m)\tilde{h}_+(\xi',\xi_m),$$
$$\tilde{U}_-(\xi',\xi_m)=\omega^{-1}(\xi',\xi_m)\sigma_-^{-1}(\xi',\xi_m)\tilde{h}_-(\xi',\xi_m)$$
$$=\left(\frac{\xi_m-i|\xi'|}{\xi_m+i|\xi'|}\right)^{-æ}\sigma_-^{-1}(\xi',\xi_m)\tilde{h}_-(\xi',\xi_m).$$

These functions $\tilde{U}_\pm$ belong to the spaces $A(\mathbb{R}^m), B(\mathbb{R}^m)$ respectively but the function $\tilde{U}_-(\xi',z)$ has a pole of order $-æ$ in the point $-i|\xi'|$. To exclude such possibility we'll use the expansion of the Cauchy type integral $\tilde{h}_-$ like [1]:

$$\tilde{h}_-(\xi',\xi_m)=(\Pi_-\tilde{h})(\xi',\xi_m)=i\sum_{k=0}^{|æ|}\Lambda_+^k(\xi',\xi_m)\Pi'\Lambda_+^{-k-1}\tilde{h}+\Lambda_+^{-æ}(\xi',\xi_m)\Pi_-\Lambda_+^{æ}\tilde{h},$$

where

$$\Lambda_+(\xi',\xi_m)=\xi_m+i|\xi'|,\quad (\Pi'\tilde{h})(\xi')\equiv\int_{-\infty}^{+\infty}\tilde{h}(\xi',\xi_m)d\xi_m$$

and notation $\Lambda_+\tilde{h}$ denotes a product of two functions $\Lambda_+(\xi',\xi_m)$ and $\tilde{h}(\xi',\xi_m)$.

Thus the pole in the point $-i|\xi'|$ will disappear if the following conditions hold:

$$\int_{-\infty}^{+\infty}\frac{\tilde{h}(\xi',\eta_m)d\eta_m}{(\eta_m+i|\xi'|)^{k+1}}=0\quad for\quad a.a.\ \xi'\in\mathbb{R}^{m-1},\quad k=0,1,\cdots,|æ|.$$

Taking into account that

$$\tilde{h}(\xi',\xi_m)=\sigma_+^{-1}(\xi',\xi_m)\sigma_{\mathcal{A}}^{-1}(\xi',\xi_m)\tilde{V}(\xi)$$

and substituting it in obtained conditions we have

$$\int_{-\infty}^{+\infty}\frac{\sigma_+^{-1}(\xi',\eta_m)\sigma_{\mathcal{A}}^{-1}(\xi',\eta_m)\tilde{V}(\xi',\eta_m)d\eta_m}{(\eta_m+i|\xi'|)^{k+1}}=0.$$

Remark 3. If we'll enlarge the space scale and consider the Sobolev–Slobodetskii spaces H^s then we'll can omit these conditions. But since this problem will be over-determined we need some additional unknowns. These are so-called co-boundary operators or potential like operators [1]. We'll study these situations in a separate paper.

7 Variable coefficients

Here we consider a difference operator with variable coefficients in the space $L_2(\mathbb{R}^m)$:

$$(\mathcal{D}u)(x)=\sum_{|k|=0}^{+\infty}p_k(x)u(x+\gamma_k),\quad x\in\mathbb{R}^m,\quad \gamma_k=(\gamma_{k_1},\cdots,\gamma_{k_m})\in\mathbb{R}^m.\quad (7.1)$$

Definition 4. The function

$$\sigma_{\mathcal{D}}(x,\xi) = \sum_{|k|=0}^{+\infty} p_k(x) e^{i\gamma_k\cdot\xi}$$

is called a symbol of the operator $\mathcal{D}$.

We'll suppose in this section that all functions $p_k(x), |k| = 0, 1, \cdots$ are continuous in $\mathbb{R}^m$, denote

$$\|p_k\|_{C(\dot{\mathbb{R}}^m)} = p_k,$$

assuming

$$\sum_{|k|=0}^{+\infty} p_k = A < +\infty.$$

Lemma 2. *The operator $\mathcal{D}$ is a linear bounded operator $L_p(\mathbb{R}^m) \longrightarrow: L_p(\mathbb{R}^m)$, $1 \le p \le +\infty$.*

Proof. We have

$$|(\mathcal{D}u)(x)| \le \sum_{|k|=0}^{+\infty} |p_k(x)|\,|u(x+\gamma_k)| \le \sum_{k=0}^{+\infty} p_k |u(x+\gamma_k)|$$

and then

$$\|\mathcal{D}u\|_1 = \int_{\mathbb{R}^m} |(\mathcal{D}u)(x)|dx \le \sum_{k=0}^{+\infty} p_k \int_{\mathbb{R}^m} |u(x+\gamma_k)| \le A\|u\|_1,$$

i.e. $\mathcal{D} : L_1(\mathbb{R}^m) \longrightarrow: L_1(\mathbb{R}^m)$ is a linear bounded operator.

In the same way we conclude that $\mathcal{D} : L_\infty(\mathbb{R}^m) \longrightarrow: L_\infty(\mathbb{R}^m)$. Further we can apply simplest interpolation theorems [10, 18] and conclude that the operator $\mathcal{D} : L_p(\mathbb{R}^m) \longrightarrow: L_p(\mathbb{R}^m)$ is a linear bounded operator for all $1 \le p \le +\infty$. □

7.1 Local principle

For every operator $\mathcal{D}$ we define an operator family $\{\mathcal{D}_{x_0}\}_{x_0\in\dot{\mathbb{R}}^m}$ where

$$\mathcal{D}_{x_0} : \; u(x) \longrightarrow \sum_{|k|=0}^{+\infty} p_k(x_0) u(x+\gamma_k), \quad x \in \mathbb{R}^m$$

and the function

$$\sigma(x_0,\xi) = \sum_{|k|=0}^{+\infty} p_k(x_0) e^{i\gamma_k\cdot\xi}$$

will be its symbol (see (2.2)) because the operator $\mathcal{D}_{x_0}$ is an operator with constant coefficients. Following Simonenko [9] we call the operator $\mathcal{D}_{x_0}$ a local representative of the operator $\mathcal{D}$ in the point x_0.

Let $\mathfrak{L}(\mathcal{D})$ be a space of linear bounded operators of type (7.1) with operator norm $\|\mathcal{D}\|_{L_p(\mathbb{R}^m)\to L_p(\mathbb{R}^m)}, 1 \le p \le +\infty$. Thus we can introduce the operator function $\mathcal{D}(x) \equiv \mathcal{D}_x$ for all $x \in \dot{\mathbb{R}}^m$.

We remind that an operator $\mathcal{D}$ has a Fredholm property if

$$Ind\ A \equiv \dim Ker\ A - \dim Coker\ A$$

is a finite number. A Fredholm property is stable under compact and small perturbations, and $Ind\ A$ is a homotopic invariant of the operator A [5].

Lemma 3. *Operator $\mathcal{D}$ has a Fredholm property in the space $L_p(\mathbb{R}^m)$ iff the family $\{\mathcal{D}_{x_0}\}_{x_0\in\dot{\mathbb{R}}^m}$ consists of invertible operators.*

Proof. Let $\mathfrak{D}$ be an algebra of difference operators of type (7.1), and $\mathfrak{P}$ be an algebra of pseudo differential operators with symbols $p(x,\xi) \in C(\dot{\mathbb{R}}^m \times \dot{\mathbb{R}}^m)$. Obviously $\mathfrak{D} \subset \mathfrak{P}$. Further let $\mathcal{P}(x)$ be a corresponding operator function for a pseudo differential operator $\mathcal{P} \in \mathfrak{P}$. We say that operator function is invertible if exists an operator function $\mathcal{P}^{-1}(x)$ such that

$$\mathcal{P}(x)\mathcal{P}^{-1}(x) = I, \quad \forall x \in \dot{\mathbb{R}}^m.$$

Now our main goal is to show that given operator function $\mathcal{D}(x)$ one can construct a pseudo differential operator $\mathcal{P} \in \mathfrak{P}$ such that $\mathcal{D}\mathcal{P}$ and $\mathcal{P}\mathcal{D}$ can be represented in the form $I + T$, where T is a compact operator. It will imply that the operator $\mathcal{D}$ has a two-sided regularizer, and thus will have a Fredholm property [5].

In the algebra $\mathfrak{P}$ we'll extract a special type of operators so called operators of a local type [9]. Without going into details let's say that pseudo differential operator from $\mathcal{P} \in \mathfrak{P}$ will be an operator of a local type if its kernel

$$K_{\mathcal{P}}(x,y) = F^{-1}_{\xi\to y} p(x,\xi)$$

is at least a continual function of variables (x,y) and generates a linear bounded operator of the type

$$u(x) \longmapsto \int_{\mathbb{R}^m} K_{\mathcal{P}}(x,x-y)u(y)dy. \tag{7.2}$$

Further we'll correct a symbol $p(x,\xi) \in C(\dot{\mathbb{R}}^m \times \dot{\mathbb{R}}^m)$ by small perturbations so that the corrected pseudo differential operator $\mathcal{P}_\varepsilon$ with a symbol $p_\varepsilon(x,\xi)$ will generate the continual kernel $K_{\mathcal{P},\varepsilon}(x,\xi)$. We need two steps.

First if $p(\infty) = c$ then we consider

$$p_1(x,\xi) = p(x,\xi) - c,$$

so that $p_1(\infty) = 0$, and $p(x,\xi) = p_1(x,\xi) + c$. It means that the operator with the symbol $p(x,\xi)$ is represented in the form $\mathcal{P}_1 + cI$.

Second one can approximate the symbol $p_1(x,\xi)$ by symbols $p_{1,\varepsilon}(x,\xi) \in S(\mathbb{R}^m \times \mathbb{R}^m)$ (Schwartz class) with smooth kernel $K_{\mathcal{P}_{1,\varepsilon}}(x,y)$. Such operator

$$(\mathcal{P}_\varepsilon u)(x) = cu(x) + \int_{\mathbb{R}^m} K_{\mathcal{P}_{1,\varepsilon}}(x,x-y)u(y)dy$$

is an operator of a local type, and

$$\|\mathcal{P}-\mathcal{P}_\varepsilon\|_{L_p(\mathbb{R}^m)\longrightarrow:L_p(\mathbb{R}^m)} \le \varepsilon.$$

It's well known [5, 9] that the family $\{\mathcal{P}_\varepsilon(x)\}$ of operators of a local type reconstructs the operator $\mathcal{P}_\varepsilon$ up to compact summand. Moreover now we can assert the following. If the operator function $\mathcal{P}_\varepsilon(x)$ is invertible then the operator $\mathcal{P}_\varepsilon^{-1}$ with the symbol $p_\varepsilon^{-1}(x,\xi)$ will be two-sided regularizer for the operator $\mathcal{P}_\varepsilon$ and consequently for $\mathcal{P}$. $\square$

Corollary 2. Operator $\mathcal{D}: L_p(\mathbb{R}^m) \to L_p(\mathbb{R}^m)$ has a Fredholm property iff

$$\sigma_{\mathcal{D}}(x,\xi) \neq 0, \quad \forall x,\xi \in \dot{\mathbb{R}}^m.$$

Lemma 4. *Index of a Fredholm operator $\mathcal{D}: L_p(\mathbb{R}^m) \to L_p(\mathbb{R}^m)$ is equal to 0.*

Proof. According to convexity of $\dot{\mathbb{R}}^m$ one can easily construct a homotopy

$$\mathcal{D}_{t,x_0} \equiv \mathcal{D}_{(1-t)x_0+tx}, \quad t \in [0,1],$$

where $x_0 \in \dot{\mathbb{R}}^m$ is an arbitrary fixed point. Since all intermediate operators are invertible then an initial operator $\mathcal{D} = \mathcal{D}_{1,x_0}$ has vanishing index. $\square$

7.2 Half-space case

This section is connected to operators in $\mathcal{D}: L_2(\mathbb{R}^m_+) \to L_2(\mathbb{R}^m_+)$. These construction are more complicated but a general idea is the same. The main result for this case is the following.

Theorem 4. *Operator $\mathcal{D}: L_2(\mathbb{R}^m_+) \to L_2(\mathbb{R}^m_+)$ has a Fredholm property iff the following conditions*

$$1)\ \sigma_{\mathcal{D}}(x,\xi) \neq 0, \quad \forall x \in \overline{\mathbb{R}^m_+}, \quad \xi \in \dot{\mathbb{R}}^m, \quad 2) \int_{-\infty}^{+\infty} d\arg\sigma_{\mathcal{D}}(\cdot,\cdot\xi_m) = 0.$$

hold.

Sketch of proof. In this case also a local principle plays key role. But main lemma describing the local principle will be distinct. Namely for our difference operator $\mathcal{D}$ the operator function will consist of two parts. The first part $\mathcal{D}_1(x)$ will be defined on $\overline{\mathbb{R}^m_+}$ and it generated by operators of type (7.2); more precisely it consists of pseudo differential operators with symbols $\sigma_{\mathcal{D}}(x,\xi$:

$$u(x) \longmapsto F^{-1}_{\xi\to x}\sigma_{\mathcal{D}}(x,\xi)\tilde{u}(\xi).$$

The second part $\mathcal{D}_2(x)$ consists of operators of the following type

$$u(x) \longmapsto \int_{\mathbb{R}^m_+} K_{\mathcal{D}}(x,x-y)u(y)dy,$$

or in more general form and in Fourier image

$$\tilde{u}(\xi) \longmapsto \sigma_{\mathcal{A}}(x,\xi)(\Pi_+\tilde{u})(x,\xi) + \sigma_{\mathcal{B}}(x,\xi)(\Pi_-\tilde{u})(x,\xi).$$

(We'll remind here that for the operator $\mathcal{D} : L_2(\mathbb{R}^m_+) \to L_2(\mathbb{R}^m_+)$ one should put in the last formula $\sigma_{\mathcal{A}}(x,\xi) = \sigma_{\mathcal{D}}(x,\xi), \sigma_{\mathcal{B}}(x,\xi) = 1$). The operator function $\mathcal{D}_2(x)$ will be defined for the points $x \in \dot{\mathbb{R}}^{m-1}$, i.e for boundary points of $\mathbb{R}^m_+$. Further we should prove that operator $\mathcal{D}$ has a Fredholm property iff both parts $\mathcal{D}_1(x)$ and $\mathcal{D}_2(x)$ are invertible for all admissible x. If so then the condition 1 in the theorem is an invertibility condition for the first operator family, and the condition 2 is the same for the second part. $\square$

Conclusions

There are many other interesting cases for studying solvability for difference equations in special canonical domain like multidimensional cones. This will be the object of another paper.

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