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A NEW MODEL OF RESOURCE PLANNING FOR OPTIMAL PROJECT SCHEDULING

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Abstract. It is well known that not only the classical model of PERT/CPM but its later improvements treat a resource planning in the project scheduling in a very limited way. Using them it is possible to calculate the optimum amount of resources taken from outside or financial expenses for separate operations but it is quite impossible to share internal resources between parallel operations. Instead of these models a new one is introduced. It combines relationships concerning sharing resources-capacities and works dynamics and perhaps other ones that express the use of materials and funds, inventory control and so on. Non-strict work precedence conditions may be used as well. The model as a whole slightly differs from the model of resource planning in complex industrial systems proposed by the author and retains its general properties, notably the form of optimality conditions. A decomposition method of the project optimum fulfillment is proposed.

Key words: project, scheduling, work, precedence, resource-capacity, control, optimum, decomposition

1. Introduction

A few approaches to project scheduling with the use of deterministic models and methods are known:

1. Critical path technique (PERT/CPM): resources-capacities are a priori attached to works; waste of materials is not considered at all; the sequence of works using the same capacity is predetermined;
2. Optimization of funds usage for temporary leasing of capacities (Kelley and Walker 1959, Fulkerson 1961, see [3]);
3. Local continuous-valued optimization of resources usage;
4. Discrete optimization of resources usage [5];
5. Heuristic approaches based on the idea of work priorities;

6. Local optimization with optimum control methods (Zimin, Ivanilov, Petrov 1971-1973 [5]).

Instead of these models we propose a new one, which treats the project fulfillment as a discrete-continuous process. Models of such kind were introduced recently for various domains [8, 9]. Their methods are based on the optimum control theory of discontinuous systems [1]. This form of a model enables the development of optimum control methods for solving the problem of sharing capacities among simultaneous works probably taking into account some other aspects, such as the optimum use of materials and inventory control.

2. The Model Formalization

First we list main features of the project scheduling problem. The project is decomposed into a set of n works. There is a relation of precedence connecting pairs of works. Resources-capacities are shared between several works. Works may be done with a varying intensity. Material requirements for the work fulfillment may be significant; these materials may be supplied continuously with a limited rate or discretely at a given set of instants and stored. Each i-th work has the set of immediately preceding works I_i divided into two subsets of works with strict and non-strict precedence, respectively I_{i1} and I_{i2}. For beginning of the i-th work the termination of all the works from I_{i1} is necessary; for the works from I_{i2} it is necessary that a certain amount of each is fulfilled.

The current work state and its change in time have the following characteristics: quantitative x_i – the amount of work done (in units of time or other corresponding to the work type); qualitative d_i: 0 – "not begun", because some of preceding works are not terminated; 1 – "in performance", 2 – "terminated"; the total amount of the work is denoted by x_{Ti}.

The period is divided into N (non-fixed number) stages with events of works beginning/termination (or some other types of events). We add the argument k for all values belonging to k-th stage. For the k-th stage the time interval is denoted as $[T(k), T(k+1)=T(k)+t(k))$, vectors of qualitative state $d(k) \in A_D$ (where A_D is a finite set) and of control $u(k) \in R^m$ are constant; the state of works (depots) is represented with the state vectors: initial $x^0(k) = x(T(k), k)$ and final $x^1(k) = x(T(k+1), k)$.

The base model regarded here is given as follows: within each stage the works in operation are fulfilled at a constant rate

$$\frac{dx_i(t,k)}{dt} = u_i(k),$$
$$0 < u_{\min i} \le u_i(k) \le u_{\max i}, \; i \in I_1(k), \;\; u_i(k) = 0, \; i \notin I_1(k). \tag{2.1}$$

Here we denote sets of works $I_l(k) = \{i \in \{1, \ldots, n\} \mid d_i(k) = l\}$ for $l = 0, 1, 2, 3$. N_R types of capacities are shared between works in performance.

$$\sum_{i \in I_l} u_i(k) \le u_{Rl}, \quad l = 1, \ldots, N_R. \tag{2.2}$$

The relationship between $x_i^1(k)$ and $x_i^0(k)$, $u_i(k)$ has the form of difference equations:

$$x_i^1(k) = x_i^0(k) + u_i(k)t(k); \tag{2.3}$$

the final state for k-th stage and initial stage for $(k+1)$-th stage are the same:

$$x_i^0(k+1) = x_i^1(k). \tag{2.4}$$

The condition of the stage termination from which the set of terminated works $S(k) \subseteq \{1,\dots,n\}$ is determined is given by

$$x_i^1(k) = x_{Ti}, \;\; i \in S(k) \subseteq I_1(k), \quad x_i^1(k) < x_{Ti}, \;\; i \in I_1(k) \setminus S(k). \tag{2.5}$$

The change of qualitative states of works at the beginning of $(k+1)$-th stage is defined as

$$d_i(k+1) = \begin{cases} 2 \;\text{ if }\; i \in S(k), \\ 1 \;\text{ if }\; d_i(k) = 0 \& I_2(k) \cup S(k) \supseteq I_{i1}. \end{cases} \tag{2.6}$$

The project ends if $I_1(N) \subset \{1,\dots,n\}$ and $I_1(N+1) = \{1,\dots,n\}$. The target functional is

$$T(N+1) \to \min. \tag{2.7}$$

Let us consider some generalizations of the base model (2.1)–(2.7). Taking into account the usage of materials we get a more general equation instead of (2.3):

$$x_i^1(k) = x_i^0(k) + f_i\big(u_i(k), u_{i1}(k), \dots, u_{iK}(k)\big)t(k),$$

where $u_{i1}(k),\dots,u_{iK}(k)$ are intensities of materials usage. We may introduce as well the dynamics of materials storage

$$x_{Mj}^1(k) = x_{Mj}^0(k) + \big(u_{Mj}(k) - u_{1j}(k) - \ldots - u_{nj}(k)\big)\, t(k),$$

where $u_{Mj}(k)$ is the intensity of j-th material supply. If the material supply is a series of batches of the amount x_{Mbj} received either in the fixed instants or when the storage contents reduces to the threshold value, then $u_{Mj}(k) = 0$ and the event of the material income is determined by using the following conditions

$$T(k+1) = T_j\big(l_j(k)\big) \;\text{ or }\; x_{Mj}^1(k) = x_{M\min j}.$$

Then x_{Mj} and the number of received batches l_j are updated as

$$x_{Mj}^0(k+1) = x_{Mj}^1(k) + x_{Mbj}, \quad l_j(k+1) = l_j(k) + 1.$$

In all other cases we have

$$x_{Mj}^0(k+1) = x_{Mj}^1(k), \quad l_j(k+1) = l_j(k).$$

For the case of materials with a permanent supply the following condition must be satisfied:

$$x_{M\min j} \le x_{Mj}^1(k) \le x_{M\max j}.$$

In many cases the project is assumed as containing passive processes with fixed durations interrelated by precedence conditions with other works and not demanding any resources. For such a kind of processes the state variable is the time from its beginning, so its dynamics and termination are determined by the equations (2.4) and

$$x_i^1(k) = x_i^0(k) + t(k), \quad x_i^1(k) = T_{Fi}.$$

The same type of processes is used to represent the condition of non-strict precedence in time between two works. The second work j may begin t_{ij} days later than the first (i-th) one. If precedence is measured as $k_{ij}x_i(t,k) - x_j(t,k)$ and the minimum precedence is x_{minij}, then the condition of achievement the necessary precedence looks like the condition of the work termination:

$$x_i^1(k) = x_{\min ij}/k_{ij}.$$

When both works are fulfilled, then the condition of precedence is defined as

$$x_j^1(k) \le k_{ij}x_i^1(k) - x_{\min ij} \quad \text{if} \quad d_i(k) = 1, \ \ d_j(k) = 1.$$

And, finally, restrictions on terms of beginning and/or termination of certain works or passive processes may exist as well as some restrictions on a duration of works.

Thus we must emphasize that all variants of the models presented here fully correspond to the general model M_0 determined below with the set of relationships (3.1)–(3.8).

3. The Process Scenario and Its Change

Let us present the main features of the model (2.1)–(2.7). The qualitative dynamics is described with difference equations (2.3) connecting initial and final states of the stage. In general they are represented as

$$x^1(k) = Y\big(d(k), x^0(k), u(k), t(k)\big) \tag{3.1}$$

where $Y(d(k), x^0(k), u(k), 0) = x^0(k)$ for any $d(k), x^0(k), u(k)$. Transformation equations (2.4) for shifts from one stage to the next one are described by

$$x^0(k+1) = X\big(S(k), d(k), x^1(k)\big). \tag{3.2}$$

A general formulation of the condition for a stage termination with a certain set of events $S(k) \subseteq \{1, \dots, L\}$ is given by

$$r_{i(s)}^Y(x^1(k)) \equiv x_{i(s)}^1(k) - x_{s0} = 0, \qquad s \in S(k), \tag{3.3}$$

it can be applied for any type of the event s with the only state variable $x_{i(s)}$ increasing monotonously with respect to $t(k)$. For other types of events the following inequality holds

$$r^Y_{i(s)}\big(x^1(k)\big) < 0, \quad s \notin S(k). \tag{3.4}$$

A qualitative state vector changes at the end of each stage as a result of events

$$d(k+1) = D\big(S(k), d(k)\big). \tag{3.5}$$

There are some restrictions on the control vector depending on the value of $d(k)$, in general they can be described by

$$r^U_j\big(d(k), u(k)\big) \le 0, \quad j \in J_1\big(d(k)\big). \tag{3.6}$$

The value of the target functional depends on the final state $F_0\big(x^1(N)\big)$. Further we assume that another type of restrictions may be included in the model, namely

$$r^Y_j\big(x^1(k)\big) \le 0, \quad j \in K_0\big(d(k)\big), \tag{3.7}$$

$$r^Y_j\big(x^1(k)\big) \le 0, \quad j \in K_1\big(d(k), S(k)\big). \tag{3.8}$$

We assume further that the model (3.1)–(3.8) satisfies some general properties. They are satisfied for the base model and it is likely that the additional form of models relationships listed below do not violate them.

Condition 1. For any $d(k) \in A_D$ the set $U_0(d(k))$ of $u(k)$ satisfying (3.6) is non-empty and bounded.

Note for the base model it is sufficient to assume that

$$\sum_{i \in I_{Rj}} u_{\min i} \le u_{Rj}.$$

Condition 2. For all the $d' \in A_D$, $x' \in R^n$, $t' \ge 0$,

$$u' \in U_{0\Delta}(d') = \{u'' \in R^m \mid r^U_j(d', u'') \le \Delta, j \in J_1(d')\},$$

where $\Delta > 0$ is a constant, the functions $Y_i(d', x', u', t')$, $r^U_j(d', u')$, $r^Y_j(x')$ are defined, continuously differentiable with respect to x', u', t' and all their first order partial derivatives satisfy the Lipschitz condition

$$|g(y') - g(y)| \le K||y' - y||,$$

where $y = (x, u, t)$, $y' = (x', u', t')$ and $K > 0$.

Condition 3. For all $s = 1, \ldots, L, d' \in A_D$, the following statements are valid:

1) The inequalities

$$r^Y_{i(s)}\big(x^0(1)\big) < 0 \tag{3.9}$$

are satisfied;

2) For all $x' \in R^n$, $u' \in U_{0\,\Delta}(d')$ the function $r^Y_{i(s)}(Y(d', x', u', t))$ rises monotonously with respect to t;

3) For all $S' \subseteq \{1, \ldots, L\}$ for which $s \in S'$ and all $x' \in R^n$ satisfying

$r^Y_{i(s)}(x') = 0$ the inequality

$$r^Y_{i(s)}\big(X(S',\, d', x')\big) = r_{s0} < 0; \tag{3.10}$$

is valid;

4) For all S', $s \notin S'$ and all $x' \in R^n$

$$r^Y_{i(s)}\big(X(S',\, d',\, x')\big) = r^Y_{i(s)}(x'). \tag{3.11}$$

Each possible process of the project fulfillment is characterized with the control v, or a succession of vectors $v(k) = (u(k), t(k))$, the scenario, or a succession of sets $S = (S(1), \dots, S(N))$, the trajectory $x = (x^0(1), x^1(1), \dots, x^0(N), x^1(N))$ and the discrete trajectory $d = (d(1), \dots, d(N))$. According to (3.5) the discrete trajectory is the function of the scenario and according to (3.1) and (3.2) the trajectory is the function of the scenario and the control. Subdividing the whole set of possible processes into the sets of processes with the definite scenario we determine $V_0(S)$ as the set of all possible v where $u(k) \in U_0(d(k))$ for any k that generates the trajectory satisfying restrictions (3.3), (3.4), (3.6)–(3.8). Conditions (3.4), (3.9)–(3.11) guarantee that $r^Y_{i(s)}(x^0(k)) < 0$ for all $s = 1, \dots, L$, $k = 1, \dots, N$. So from $r^Y_{i(s)}(Y(d(k), x^0(k), u(k), t(k))) = 0$, $s \in S(k)$, we conclude that the obligatory relationship $t(k) > 0$ takes place.

But $V_0(S)$ is not a closed set and for $v^* = \lim_{r\to\infty} v^{(r)}$, $v^{(r)} \in V_0(S)$ we can say that for the corresponding x^* the inequality $r^Y_{i(s)}(x^{*1}(k)) \le 0$, $s \notin S(k)$ is valid. So we determine another model M_1 with the set of relationships (3.1)–(3.3), (3.5)–(3.8) and

$$r^Y_{i(s)}\big(x^1(k)\big) \le 0, \quad s \notin S(k). \tag{3.12}$$

Analogously for the model M_1 we conclude formally that $t(k) \ge 0$ for any k. According to the property of $Y(d(k), x^0(k), u(k), 0)$ the values of $u(k)$ for stages with $t(k)$=0 do not affect the sequence of $x^0(k)$, $x^1(k)$ for stages with $t(k) > 0$. So for any control corresponding to the scenario having $\dim S(k) > 1$ for a certain k we can use the other scenario representations. Both properties are used in the iterative search of the optimum scenario.

The other representation of the model M_1 is given by the model M_2 determined with the set of relationships (3.1)–(3.3), (3.5)–(3.8) and

$$t(k) \ge 0, \quad k = 1, \dots, N, \tag{3.13}$$

and (3.12) for $k = N$. For M_1 and M_2 the sets $V_1(S)$ and $V_2(S)$ are determined analogously to $V_0(S)$. The equivalence of both representation is asserted by the following lemma.

Lemma 1. *The following equality $V_1(S) = V_2(S)$ is valid.*

Proof. Let $V_1(S) \ne \emptyset$ and $v \in V_1(S)$. Then $t(k) \ge 0$ for any k, so $v \in V_2(S)$. Let $V_2(S) \ne \emptyset$ and $v \in V_2(S)$. If $s \in S(k)$ and $s \in S(k')$, $k' < k$, then for $k'' = k' + 1, \dots, k - 1$

$$r^Y_{i(s)}\big(x^0(k'+1)\big) \le \ldots \le r^Y_{i(s)}\big(x^0(k'')\big) \le r^Y_{i(s)}\big(x^1(k'')\big) \qquad (3.14)$$
$$= r^Y_{i(s)}\big(x^0(k''+1)\big) \le r^Y_{i(s)}\big(x^1(k''+1)\big) \le r^Y_{i(s)}\big(x^1(k)\big) = 0$$

is valid due to Condition 3. If $s \in S(k)$ and $s \notin S(k'')$ for all $k'' < k$, then for $k'' = 1, \ldots, k-1$ (3.14) is valid as well. And if $s \notin S(k'')$ for all k'' then for k''=1,..., N we get

$$r^Y_{i(s)}(x^1(k'')) = r^Y_{i(s)}(x^0(k''+1)) \le r^Y_{i(s)}(x^1(k''+1)) \le \ldots \le r^Y_{i(s)}(x^1(N)) \le 0.$$

So $r^Y_{i(s)}(x^1(k'')) \le 0$ for all k''=1,..., N and for all $s \notin S(k'')$, therefore (3.12) is valid for all the k'', thus $v \in V_1(S)$. ■

For a given scenario the set of relationships of model M_2 defines the optimization problem for a discrete-time process with known optimality conditions [1, 2, 4] and efficient numerical methods including [6, 7]. However, we are interested in the project optimization regardless of events succession.

We consider two aims related to the scenario change for a given $v \in V_2(S)$:

- First, to separate two simultaneous events sets $S_1 = S(k'-1)$ and $S_2 = S(k')$ for which $t(k') = 0$ with a short stage.
- Second, to make simultaneous two events sets $S_1 = S(k'-1)$ and $S_2 = S(k')$ initially separated with a short stage.

To reach both aims we seek to find $v_A \in V_2(S)$ for which

$$v_A(k) = v(k) + \varepsilon\delta v(k) + O(\varepsilon^2), \quad k \ne k';$$

and

$$u_A(k') = u' \in U_0(D(S(k'-1), d(k'-1))), \quad t_A(k') = \varepsilon$$

for the first aim and $u_A(k') = u(k')$, $t_A(k')$=0 for the second aim.

The set of the model restrictions for a given scenario may be represented in the following general form:

$$F_j(v,S) \le 0, \quad j \in I_1(S), \quad F_j(v,\ S) = 0, \quad j \in I_2(S). \qquad (3.15)$$

The target functional is treated as $F_0(v,S)$ as well. Let us denote (for a feasible control v and $\varepsilon \ge 0$) the set of ε–active restrictions for any $J_1 \subseteq I_1(S)$ as

$$J_{1\varepsilon}(v,S) = \{j \in J_1 | F_j(v,S) \ge -\varepsilon\}.$$

We define $I_\varepsilon(v,S)$ as $I_{1\varepsilon}(v,S) \cup I_2(S)$ and introduce obvious notation $I^Y(k,S)$ and $I^U(k,S) = J_1(d(k))$. For $J \subseteq I_1(S) \cup I_2(S)$, $v' \in V_2(S)$ we denote by $F(v',S,J)$ the vector $F_j(v',S)$, $j \in J$, and let $b_j(k;v',S) = \nabla_{v(k)}F_j(v',S)$, $B_j(v',S)$ be the vector resulting from concatenation of all $b_j(k;v',S)$, $k \ne k'$. $B(v',S,J)$ denotes the matrix which rows are $B_j(v',S)$, $j \in J$. We suppose that $B_j(v',S)$, $j \in J$, are linearly independent that is guaranteed if the following condition is satisfied.

Condition 4(regularity condition).

1) For an arbitrary $v \in V_0(S)$ vectors $F_{jv}(v, S)$, $j \in I_0(v,S)$, are linearly independent;

2) For an arbitrary $u(k)$ satisfying (3.6) vectors $F_{ju}(d(k),u(k))$, $j \in J_{10}(d(k))$, are linearly independent.

If Condition 4 is valid then ε_0 exists, such that for any $0 \le \varepsilon \le \varepsilon_0$ the regularity conditions are valid not only for 0-active restrictions but for ε-active ones as well. For the base problem (2.1)–(2.7) the regularity conditions are valid for all possible S unless the problem parameters satisfy a certain equation set.

Let $C(v',S,J)$ be a $\dim(J) \times \dim(J)$ submatrix of $B(v',S,J)$ with the minimum inverse matrix norm. The Condition 4 yields $c_{inv} > 0$ for which

$$\|(C(v',S,J))^{-1}\| \le c_{inv} \text{ for all } v' \in V_2(S), \quad J \subseteq I_\varepsilon(v,S), \quad 0 \le \varepsilon \le \varepsilon_0.$$

For both aims $v_A(k')$ satisfy the respective restrictions (3.6), (3.13). All other restrictions (3.15) will be satisfied provided that $||\delta v|| \le n_V$ if for a $\varepsilon \le \varepsilon_0/n_V$ a control v_A satisfies the equations set for the given v:

$$G_j(v_A,S) \equiv F_j(v_A,S) - F_j(v,S) = 0, \; j \in I'_\varepsilon = I_\varepsilon(v,S) \setminus J_{1\varepsilon}(d(k')).$$

We propose a Newton-like method of its solution with initial $v^{(0)}$ where $v^{(0)}(k) = v(k)$, $k \ne k'$, $v^{(0)}(k') = v_A(k')$ and recursive relationships

$$B\big(v^{(r)},S,I'_\varepsilon\big)\big(v^{(r+1)} - v^{(r)}\big) = -G\big(v^{(r)},S,I'_\varepsilon\big), \tag{3.16}$$

from which the vector $v^{C(r+1)}$ of $v^{(r+1)}$ components corresponding to columns of C may be determined as

$$v^{C(r)} - (C(v^{(r)},S,I'_\varepsilon))^{-1}G(v^{(r)},S,I'_\varepsilon),$$

the rest components being zeros that yields the unique solution $v^{(r+1)}$. Complying $(C(v^{(r)},S,I'_\varepsilon))^{-1}$ with zero columns to the $\dim(J) \times M$ matrix $Q(v^{(r)},S,I'_\varepsilon)$ we represent (3.16) as

$$v^{(r+1)} = v^{(r)} - Q\Big(v^{(r)},S,I'_\varepsilon\Big)G\Big(v^{(r)},S,I'_\varepsilon\Big),$$

thus

$$G_j\big(v^{(r+1)},S\big) = \mathcal{O}\big(\|G(v^{(r)},I'_\varepsilon)\|^2\big), \quad j \in I'_\varepsilon,$$
$$\|G\big(v^{(r+1)},S,I'_\varepsilon\big)\| = \mathcal{O}\big(\|G(v^{(r)},S,I'_\varepsilon)\|^2\big),$$

hence the iteration process (3.16) converges superlinearly if ε is sufficiently small and $\|(C(v^{(r)},S,I'_\varepsilon))^{-1}\| \le c_{inv}$ for all r. In that case we can write

$$v^* = \lim_{r\to\infty} v^{(r)}; \quad \|v^* - v^{(1)}\| \le K^*\big(c_{inv}\,\|G\big(v^{(1)},S,I'_\varepsilon\big)\|\big)^2 \le K^{**}\varepsilon^2.$$

To determine $F_{jv}(v', S)$ we can use the formula for $j \in \{0\} \cup I^Y(k_j, S)$ (see, [4]):

$$\delta F_j = \big(p_j^1(S,k'), \delta x^1(k')\big) + \sum_{k=k'+1}^{k_j} \big(p_j^0(S,k), Y_v\big(d(k), x^0(k), v(k)\big)\delta v(k)\big), \quad (3.17)$$

where $k_0 = N$ and for conjugate variables $p_j^0(S,k')$, $p_j^1(S,k')$ we have:

$$p_j^1(S,k_j) = \big(r_{jx}^Y\big(x^1(k_j)\big)\big)^T, \; p_j^0(S,k)Y_x^T\big(d(k), x^0(k), v(k)\big)\, p_j^1(S,k),$$
$$p_j^1(S,k-1) = X_x^T\big(S(k-1), d(k-1), x^1(k-1)\big)\, p_j^0(S,k), \quad k = k_j, \ldots, 1.$$

From (3.17) we have for $j \in I^Y(k,S)$ and $j \in I^U(k,S)$

$$b_j(v,S,k) = Y_v^T\big(d(k), x^0(k), v(k)\big) p_j^0(S,k),$$
$$b_j(v,S,k) = \big(r_{u(1)}^U\big(d(k), u(k)\big), \ldots, r_{u(m)}^U\big(d(k), u(k)\big), 0\big).$$

In the case of the first and second aims we have for $F_j(v^{(0)}, S)$, $j \in \{0\} \cup I^Y(k', S)$ the following formulas

$$F_j(v,S) + \big(p_j^1(S,k'), Y_t\big(d(k'), x^0(k'), u_A(k'), 0\big)\big)\, t_A(k') + \mathcal{O}\big(t_A^2(k')\big),$$
$$F_j(v,S) - \big(p_j^1(S,k'), Y_t\big(d(k'), x^0(k'), v(k')\big)\big)\, t(k') + \mathcal{O}\big(t^2(k')\big). \quad (3.18)$$

Then from (3.18) we have

$$F_0(v^*, S) = F_0(v,S) + \varepsilon\big(q_{00}(v,S),\; Y_t\big(d(k'), x^0(k'), u_A(k'), 0\big)\big) + \mathcal{O}(\varepsilon^2), \quad (3.19)$$

where

$$q_{00}(v,\; S) = p_0^1(S,k') - F_{ov}(v,\; S) \sum_{j \in I_0^Y(v,S)} Q_j\big(v^{(0)}, S, I_0'\big) p_j^1(S,k').$$

From (3.19) we get the necessary optimality condition formulated in the [8].

Theorem 1. *If the pair* $(S, v \in V_2(S))$ *gives the solution of the problem* (3.1)–(3.8) *and for some* k' *we have* $\dim(S(k'-1)) > 1$*, then for any* S_A *for which*

$$S_A(k) = S(k), \;\; k < k'-1, \quad S_A(k'-1) \cup S_A(k') = S(k'-1),$$
$$S_A(k) = S(k-1), \quad k = k'+1, \ldots, N+1,$$

there exists a vector $q_{00}(v, S_A)$ *such that for any* $u_A(k') \in U_0(d_A(k'))$ *the following inequality is valid*

$$\big(\, q_{00}(v, S_A),\; Y_t(d(k'), x^0(k'), u_A(k'), 0)\,\big) \geq 0.$$

4. Transformation of Optimality Conditions and Numerical Method Based on Decomposition with Respect to Restrictions Set

To make the use of both types of optimality conditions more convenient we represent control variations with a decomposition scheme as

$$\delta v = H_1 y_1 + H_2 y_2 + \ldots + H_Q y_Q, \tag{4.1}$$

where the constraints set $J \supseteq I_\varepsilon(v, S)$ is shared into Q subsets, $J_1, \ldots, J_Q$, and matrices $H_1, \ldots, H_Q$ are determined from the condition: for any δv

$$(F_{iv}(v,S), \delta v) = (F_{iu}(v,S),\ H_q y_q),\ \ i \in J_q.$$

With the use of decomposition the usual Zoutendijk type necessary optimality conditions [4] are transformed into the following form:

Theorem 2. *If vector v is the solution of (3.15), the regularity condition holds for $J \supseteq I_0(v,S)$ and the set of matrices $H_1, \ldots, H_Q$ determines a decomposition scheme on $(v,\ J)$, then for any $q = 1, \ldots, Q$ for arbitrary y_q satisfying inequalities*

$$(F_{iv}(v,S), H_q y_q) \le 0, \quad i \in J_q \cap I_{10}(v,S),$$

$$(F_{iv}(v,S), H_q y_q) = 0, \quad i \in J_q \cap I_2(S),$$

the inequality is valid

$$(F_{0v}(v,S), H_q y_q) \ge 0. \tag{4.2}$$

Constructing an altered scenario we use the representation

$$y_q = \sum_{j \in J_q} \varphi_j \cdot c_{qj}$$

where vectors c_{qj}, $j \in J_q$, are determined from the equations

$$F_{jv}^T(v,S)\, H_q c_{qj} = 1, \quad F_{iv}^T(v,S)\, H_q c_{qj} = 0, \quad j, i \in J_q, \quad i \ne j. \tag{4.3}$$

The relationship (4.1) then becomes the set of N relationships of the form

$$\delta v(k) = H_1(k) y_1 + \ldots + H_Q(k) y_Q \tag{4.4}$$

and the iteration process (3.16) is transformed into

$$v^{(r+1)}(k) = v^{(r)}(k) - \sum_{q=1}^{Q} \Big[\sum_{j \in J_q} G_j(v^{(r)}, S, I_0') \cdot H_{qj}^{(r)}(k) \cdot c_{qj}^{(r)} \Big].$$

The computation of the optimum control may be based on the generalization of the direct decomposition method combining features of feasible directions and gradient-restoration methods [6, 7]. In the proposed method for most iterations optimization within a fixed scenario the following calculations are performed:

- For $q = 1, \dots, Q$) the following problems are solved with respect to a scalar η_{0q} and a vector y_q for a given δ:

$$\eta_{0q} = (F_{0v}(v, S),\ H_q y_q) \to \min, \tag{4.5}$$

$$F_i(v, S) + (F_{iv}(v, S),\ H_q y_q) \le 0, \qquad i \in J_q \cap I_{1\delta}(v, S), \tag{4.6}$$

$$(F_{iv}(v, S),\ H_q y_q) = 0, \qquad i \in J_q \cap I_2(S), \tag{4.7}$$

$$-1 \le y_{qi} \le 1, \qquad i = 1, \dots, M_q, \tag{4.8}$$

- If

$$\eta_0 = \eta_{01} + \dots + \eta_{0Q} > -c_\eta \delta^\gamma,$$

then δ is diminished (e.g., $\delta = \delta/2$) and problems (4.5)–(4.8) are solved once more. Otherwise the next iteration control

$$v' = v + \alpha(H_1 y_1 + H_2 y_2 + \dots + H_Q y_Q)$$

is taken.
- If for any k' we get $\dim(S(k')) > 1$, then for some iterations the opportunity of changing from original scenario S to S_A is tested by solution of the problem

$$\big(q_{00}(v, S_A),\ Y_t(d(k'), x^0(k'), u_A(k'), 0)\big) \to \min, \quad u_A(k') \in U_0(d_A(k')). \tag{4.9}$$

- If the target function in (4.9) is less than $\eta_0 < 0$ for a preceding iteration, then the scenario is changed as it was described above.

Since there is a finite number of shifts to another scenario, the results of the method convergence stay valid, but we get more than the optimum for the last scenario.

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