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Existence of Multiple Positive Solutions for Quasilinear Elliptic Systems Involving Critical Hardy–Sobolev Exponents and Sign-Changing Weight Function

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Abstract. In this paper, we consider a class of quasilinear elliptic systems with weights and the nonlinearity involving the critical Hardy–Sobolev exponent and one sign-changing function. The existence and multiplicity results of positive solutions are obtained by variational methods.
Keywords: multiple positive solutions, Nehari manifold, critical Hardy–Sobolev exponent.

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1 Introduction

The aim of this paper is to establish the existence and multiplicity of nontrivial non-negative solutions to the quasilinear elliptic system

$$\begin{cases} -\mathrm{div}\big(|x|^{-ap}|\nabla u|^{p-2}\nabla u\big) = \dfrac{1}{p^*|x|^{bp^*}}F_u(x,u,v) + \lambda f(x)\dfrac{1}{|x|^{\beta}}|u|^{q-2}u, & \text{in } \Omega, \\ -\mathrm{div}\big(|x|^{-ap}|\nabla v|^{p-2}\nabla v\big) = \dfrac{1}{p^*|x|^{bp^*}}F_v(x,u,v) + \mu f(x)\dfrac{1}{|x|^{\beta}}|v|^{q-2}v, & \text{in } \Omega, \\ u>0,\ v>0, & \text{in } \Omega, \\ u=v=0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $0 \in \Omega$ is a bounded domain in $\mathbb{R}^N$ $(N \geq 3)$ with the smooth boundary $\partial\Omega$, $F \in C^1(\overline{\Omega}\times(\mathbb{R}^+)^2, \mathbb{R}^+)$ is positively homogeneous of degree p^*. Here $p^* = p(a,b) \triangleq \frac{pN}{N-p(1+a-b)}$ is the Hardy–Sobolev critical exponent. Note that $\mathbf{2}(a,a) = \frac{2N}{N-2} = 2^*$ is the Sobolev critical exponent. Thus $F(x,tu,tv) = t^{p^*}F(x,u,v)$ $(t>0)$ hold for all $(x,u,v) \in \overline{\Omega}\times(\mathbb{R}^+)^2$, $(F_u(x,u,v), F_v(x,u,v)) = \nabla F(x,u,v)$.

We make the following assumptions:

(A1) $\beta < (1+a)p_1 + N\left(1 - \frac{p_1}{p}\right)$, with $1 < q < p^*$, $1 < p_0 \le \frac{Np}{N-p}$ and $q < p_1 < \frac{Np}{N-p}$ such that $\frac{1}{p_0} + \frac{q}{p_1} = 1$, $(\lambda,\mu) \in \mathbb{R}^2 \setminus \{(0,0)\}$ and $f(x) \in L^{p_0}(\Omega, |x|^{-\beta})$, with $f^{\pm}(x) = \max\{\pm f, 0\} \ne 0$.

Problem (1.1) is related to the well known Caffarelli–Kohn–Nirenberg inequality in [12, 22],

$$\left(\int\limits_{\mathbb{R}^N} |x|^{-bp^*}|u|^{p^*}\,dx\right)^{\frac{p}{p^*}} \le C_{a,b} \int\limits_{\mathbb{R}^N} |x|^{-ap}|\nabla u|^p\,dx, \text{ for all } u \in C_0^\infty\left(\mathbb{R}^N\right), \quad (1.2)$$

where $1 < p < N$, $-\infty < a < \frac{N-p}{p}$, $a \le b \le a+1$, $p^* = \frac{Np}{N-p(1+a-b)}$.

If $1 < p < N$ and $-\infty < a < \frac{N-p}{p}$ we denote by $W_0^{1,p}(\Omega, |x|^{-ap})$ the completion of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\| = \left(\int_\Omega |x|^{-ap}|\nabla u|^p\,dx\right)^{1/p}.$$

By using the inequality (1.2) and the boundedness of Ω, was proved in [22] that there exists $C > 0$ such that

$$\left(\int_\Omega |x|^{-\delta}|u|^r\,dx\right)^{p/r} \le C\int_\Omega |x|^{-ap}|\nabla u|^p\,dx, \quad \text{for all } u \in W_0^{1,p}\left(\Omega, |x|^{-ap}\right),$$

where $1 \le r \le \frac{Np}{N-p}$, $\delta \le (a+1)r + N[1-(r/p)]$, which is said the Caffarelli–Kohn–Nirenberg's inequality. In other words, the embedding $W_0^1(\Omega, |x|^{-ap}) \hookrightarrow L^r(\Omega, |x|^{-\delta})$ is continuous if $1 \le r \le \frac{Np}{N-p}$ and $\delta \le (a+1)r + N[1-(r/p)]$. Moreover, this embedding is compact if $1 \le r < \frac{Np}{N-p}$ and $\delta < (a+1)r + N[1-(r/p)]$, see Theorem 2.1 in [22] for the case when $\nu = 0$.

Now, we define the space $W = W_0^{1,p}(\Omega, |x|^{-ap}) \times W_0^{1,p}(\Omega, |x|^{-ap})$ with the norm

$$\|(u,v)\| = \left(\int_\Omega |x|^{-ap}|\nabla u|^p\,dx + \int_\Omega |x|^{-ap}|\nabla v|^p\,dx\right)^{1/p}.$$

Also, we can define the best Hardy–Sobolev constant:

$$A = \mathbf{A}_{a,b}(\Omega) = \inf_{u \in W_0^{1,p}(\Omega,|x|^{-ap})\setminus\{0\}} \frac{\int_\Omega |x|^{-ap}|\nabla u|^p\,dx}{\left(\int_\Omega |x|^{-bp^*}|u|^{p^*}\,dx\right)^{\frac{p}{p^*}}}.$$

In recent years, several authors have used the Nehari manifold to solve semilinear and quasilinear problems (see [1, 2, 6, 7, 8, 9, 10, 16, 20] and references therein). Brown and Zhang [11] have studied a subcritical semi-linear elliptic equation with a sign-changing weight function and a bifurcation real parameter in the case $p = 2$ and Dirichlet boundary conditions. Exploiting the relationship between the Nehari manifold and fibering maps (i.e., maps of the form $t \mapsto J_\lambda(tu)$ where J_λ is the Euler functional associated with the equation), they gave an interesting explanation of the well-known bifurcation result. In fact, the nature of the Nehari manifold changes as the parameter λ crosses the bifurcation value. In this work, we give a variational method which is similar to the fibering method (see [14] or [6, 11]) to prove the existence of at least two nontrivial nonnegative solutions of problem (1.1). Some authors also studied the singular problems with Hardy–Sobolev critical exponents ([3, 17, 18] the references therein).

Before stating our results, we need the following assumptions:

(H1) $F : \overline{\Omega} \times (\mathbb{R}^+) \times (\mathbb{R}^+) \to \mathbb{R}^+$ is a C^1 function such that $F(x, tu, tv) = t^{p^*} F(x, u, v)$ $(t > 0)$ hold for all $(x, u, v) \in \overline{\Omega} \times (\mathbb{R}^+)^2$;

(H2) $F(x, u, 0) = F(x, 0, v) = F_u(x, u, 0) = F_v(x, 0, v) = 0$ where $u, v \in \mathbb{R}^+$;

(H3) $F_u(x, u, v)$, $F_v(x, u, v)$ are strictly increasing functions about u, v for all $u > 0$, $v > 0$.

Moreover, using assumption (H1), we have the so-called Euler identity

$$(u, v) \cdot \nabla F(x, u, v) = p^* F(x, u, v), \tag{1.3}$$

$$F(x, u, v) \le K\big(|u|^p + |v|^p\big)^{\frac{p^*}{p}}, \quad \text{for some constant } K > 0. \tag{1.4}$$

This paper is divided into three sections, organized as follows. In Section 2, we give some notations, preliminaries, properties of the Nehari manifold and set up the variational framework of the problem. In Section 3, we give our main results.

2 Preliminaries

Let us consider Ω a domain in $\mathbb{R}^N$, $0 \in \Omega$, $1 < p < N$, $0 \le a < (N-p)/p$, $a \le b < a+1$ and $p^* = \frac{pN}{N-p(1+a-b)}$. We define the space

$$W^{1,p}_{a,e}(\Omega) = \big\{u \in L^{p^*}\big(\Omega, |x|^{-\mathbf{b}p}\big)\colon |\nabla u| \in L^p\big(\Omega, |x|^{-ap}\big)\big\},$$

equipped with the norm

$$\|u\|_{W^{1,p}_{a.e}(\Omega)} := \|u\|_{L^{p^*}(\Omega, |x|^{-bp^*})} + \|\nabla u\|_{L^p(\Omega, |x|^{-ap})}.$$

We consider the constant $\widetilde{S}_{a,p}$ given by

$$\widetilde{S}_{a,p} := \inf\left\{ \frac{\int_{\mathbb{R}^N} |x|^{-ap} |\nabla u|^p \, dx}{(\int_{\mathbb{R}^N} |x|^{-bp^*} |u|^{p^*} \, dx)^{\frac{p}{p^*}}} : u \in W^{1,p}_{a.e}\big(\mathbb{R}^N\big) \setminus \{0\} \right\}.$$

Also, we define

$$R^{1,p}_{a.e}(\Omega) = \big\{u \in W^{1,p}_{a.e}(\Omega)\colon u(x) = u\big(|x|\big)\big\},$$

endowed with the norm

$$\|u\|_{R^{1,p}_{a.e}(\Omega)} = \|u\|_{W^{1,p}_{a.e}(\Omega)}.$$

Actually, Horiuchi in [15] proved that, if $a \geq 0$,

$$\widetilde{S}_{a,p,R} := \inf\left\{ \frac{\int_{\mathbb{R}^N} |x|^{-ap}|\nabla u|^p\,dx}{(\int_{\mathbb{R}^N} |x|^{-bp^*}|u|^{p^*}\,dx)^{\frac{p}{p^*}}} : u \in R^{1,p}_{a.e}(\mathbb{R}^N) \setminus \{0\} \right\} = \widetilde{S}_{a,p},$$

and it is achieved by functions of the form

$$y_\epsilon(x) := k_{a,p}(\epsilon)U_{a,p,\epsilon}(x), \quad \forall \epsilon > 0,$$

where

$$k_{a,p}(\epsilon) = \widetilde{c}\epsilon^{\frac{N-p(1+a-b)}{p^2(1+a-b)}}, \quad \text{and} \quad U_{a,p,\epsilon}(x) = \big(\epsilon + |x|^{\frac{p(1+a-b)(N-p-ap)}{(p-1)(N-p(1+a-b))}}\big)^{-\frac{N-p(1+a-b)}{p(1+a-b)}}.$$

We observe that by the Caffarelli–Kohn–Nirenberg's inequality follows that $W^{1,p}_0(\Omega, |x|^{-ap})$ is a subset of $W^{1,p}_{a.e}(\mathbb{R}^N)$, then $\widetilde{S}_{a,p} \leq A$.

We need the following lemma (the proof of this lemma follows exactly as in [19]).

Lemma 1. *Let R_1, c_1 be positive constants with $B(0,3R_1) \subset \Omega$ and $\psi \in C^\infty_0(B(0,3R_1))$ with $\psi \geq 0$ in $B(0,3R_1)$ and $\psi = 1$ in $B(0,2R_1)$, then the function given by*

$$u_\epsilon(x) := \psi(x)U_{a,p,\epsilon}(x)/\|\psi U_{a,p,\epsilon}\|_{L^{p^*}(\Omega,|x|^{-bp})},$$

satisfies

$$\|u_\epsilon\|^{p^*}_{L^{p^*}(\Omega,|x|^{-bp})} = 1, \qquad \|\nabla u_\epsilon\|^p_{L^p(\Omega,|x|^{-ap})} \leq \widetilde{S}_{a,p,R} + O\big(\epsilon^{\frac{N-p(1+a-b)}{p(1+a-b)}}\big),$$

and

$$\big\|f^{1/q}u_\epsilon\big\|^q_{L^q(\Omega,|x|^{-\beta})} \geq \begin{cases} O(\epsilon^{\frac{(N-p(1+a-b))q}{p^2(1+a-b)}}), & \text{if } q < \frac{(N-\beta)(p-1)}{N-p-ap}, \\ O(\epsilon^{\frac{(N-p(1+a-b))q}{p^2(1+a-b)}}|\ln(\epsilon)|), & \text{if } q = \frac{(N-\beta)(p-1)}{N-p-ap}, \\ O(\epsilon^{\frac{(N-p(1+a-b))(p-1)[(N-\beta)p-(N-p-ap)q]}{p^2(1+a-b)(N-p-ap)}}), & \text{if } q > \frac{(N-\beta)(p-1)}{N-p-ap}, \end{cases} \tag{2.1}$$

for all $f \in L^{p_0}(\Omega, |x|^{-\beta})$ with $f \geq 0$ for a.e. in $B(0,3R_1)$ and $\inf_{B(0,2R)} f > 0$ *for some $0 < R \leq R_1$. Moreover, the inequality (2.1) is uniform in $f \in L^{p_0}(\Omega, |x|^{-\beta})$ satisfying: $f \geq 0$ for a.e. in $B(0,3R_1)$ and*

$$\big(1 + R^{\frac{p(1+a-b)(N-p-ap)}{(p-1)(N-p(1+a-b))}}\big)^{-\frac{(N-p(1+a-b))q}{p(1+a-b)}} R^{N-\beta} \inf_{B(0,2R)} f \geq c_1,$$

for some $R \in (0, R_1]$.

Now, by (1.3) the corresponding energy functional of problem (1.1) is defined by

$$J_{\lambda,\mu}(u,v) = \frac{1}{p}\big\|(u,v)\big\|^p - \frac{1}{p^*}\int_\Omega |x|^{-bp^*}F(x,u,v)\,dx - \frac{1}{q}K_{\lambda,\mu}(u,v),$$

for each $(u,v) \in W$, where $K_{\lambda,\mu}(u,v) = \int_\Omega (\lambda f|x|^{-\beta}|u|^q + \mu f|x|^{-\beta}|v|^q)\,dx$.

In order to verify $J_{\lambda,\mu} \in C^1(W,\mathbb{R})$, we need the following lemmas.

Lemma 2. *Suppose that* (H3) *holds. Assume that* $F \in C^1(\overline{\Omega} \times \mathbb{R}^2, \mathbb{R})$ *is positively homogeneous of degree* p^*, *then* $F_u, F_v \in C(\overline{\Omega} \times {\mathbb{R}^+}^2, \mathbb{R}^+)$ *are positively homogeneous of degree* $p^* - 1$.

Moreover by the Lemma 2, we get the existence of a positive constant M such that

$$\big|F_u(x,u,v)\big| \le M\big(|u|^{p^*-1} + |v|^{p^*-1}\big), \tag{2.2}$$

$$\big|F_v(x,u,v)\big| \le M\big(|u|^{p^*-1} + |v|^{p^*-1}\big), \quad \forall x \in \overline{\Omega},\ u,v \in \mathbb{R}^+. \tag{2.3}$$

By the weighted Hardy–Sobolev inequality, (2.2) and (2.3), $J_{\lambda,\mu} \in C^1(W,\mathbb{R})$.

Now, we consider the problem (1.1) on the Nehari manifold. Define the Nehari manifold

$$N_{\lambda,\mu} = \big\{(u,v) \in W \setminus \{(0,0)\} \mid \big\langle J'_{\lambda,\mu}(u,v),(u,v)\big\rangle = 0\big\},$$

where

$$\big\langle J'_{\lambda,\mu}(u,v),(u,v)\big\rangle = \big\|(u,v)\big\|^p - \int_\Omega |x|^{-bp^*} F(x,u,v)\,dx - K_{\lambda,\mu}(u,v).$$

Note that $N_{\lambda,\mu}$ contains every nonzero solution of (1.1). Define

$$\Phi_{\lambda,\mu}(u,v) = \big\langle J'_{\lambda,\mu}(u,v),(u,v)\big\rangle,$$

then for $(u,v) \in N_{\lambda,\mu}$

$$\begin{aligned}
&\big\langle \Phi'_{\lambda,\mu}(u,v),(u,v)\big\rangle \\
&\quad = p\big\|(u,v)\big\|^p - p^* \int_\Omega |x|^{-bp^*} F(x,u,v)\,dx - qK_{\lambda,\mu}(u,v) && (2.4)\\
&\quad = (p-q)\big\|(u,v)\big\|^p - (p^*-q) \int_\Omega |x|^{-bp^*} F(x,u,v)\,dx && (2.5)\\
&\quad = (p-p^*)\big\|(u,v)\big\|^p - (q-p^*)K_{\lambda,\mu}(u,v) && (2.6)\\
&\quad = (p-p^*) \int_\Omega |x|^{-bp^*} F(x,u,v)\,dx - (q-p)K_{\lambda,\mu}(u,v). && (2.7)
\end{aligned}$$

Now, we split $N_{\lambda,\mu}$ into three parts:

$$\begin{aligned}
N^+_{\lambda,\mu} &= \big\{(\mathbf{u},\mathbf{v}) \in N_{\lambda,\mu} \colon \big\langle \Phi'_{\lambda,\mu}(u,v),(u,v)\big\rangle > 0\big\},\\
N^0_{\lambda,\mu} &= \big\{(u,v) \in N_{\lambda,\mu} \colon \big\langle \Phi'_{\lambda,\mu}(u,v),(u,v)\big\rangle = 0\big\},\\
N^-_{\lambda,\mu} &= \big\{(u,v) \in N_{\lambda,\mu} \colon \big\langle \Phi'_{\lambda,\mu}(u,v),(u,v)\big\rangle < 0\big\}.
\end{aligned}$$

To state our main result, we now present some important properties of $N^+_{\lambda,\mu}$, $N^0_{\lambda,\mu}$ and $N^-_{\lambda,\mu}$.

Lemma 3. *There exists a positive number* $\Lambda = \Lambda(q,N,K,C,|\Omega|) > 0$ *such that if*

$$0 < \big(|\lambda|\|f\|_{L^{p_0}(\Omega,|x|^{-\beta})}\big)^{\frac{p}{p-q}} + \big(|\mu|\|f\|_{L^{p_0}(\Omega,|x|^{-\beta})}\big)^{\frac{p}{p-q}} < \Lambda,$$

then $N^0_{\lambda,\mu} = \emptyset$.

Proof. Suppose opposite, that for

$$\Lambda = \left(\frac{p-q}{K(p^*-q)}\right)^{\frac{p}{p^*-p}} \left(\frac{p^*-p}{p^*-q}\right)^{\frac{p}{p-q}} C^{-\frac{p^*}{p^*-p}-\frac{q}{p-q}}$$

there exists (λ,μ) with

$$0 < \left(|\lambda|\|f\|_{L^{p_0}(\Omega,|x|^{-\beta})}\right)^{\frac{p}{p-q}} + \left(|\mu|\|f\|_{L^{p_0}(\Omega,|x|^{-\beta})}\right)^{\frac{p}{p-q}} < \Lambda,$$

such that $N^0_{\lambda,\mu} \neq \emptyset$. Then for $(u,v) \in N^0_{\lambda,\mu}$, by (2.5) and (2.6) we have

$$0 = \left\langle \Phi'_{\lambda,\mu}(u,v),(u,v)\right\rangle = (p-q)\left\|(u,v)\right\|^p - (p^*-q)\int_\Omega |x|^{-bp^*}F(x,u,v)\,dx$$
$$= (p-p^*)\left\|(u,v)\right\|^p - (q-p^*)K_{\lambda,\mu}(u,v).$$

By the Caffarelli – Kohn – Nirenberg inequality, the Minkowski inequality and estimate (1.4), one can get

$$\begin{aligned}\int_\Omega |x|^{-bp^*}F(x,u,v)\,dx &\le K\biggl(\int_\Omega |x|^{-bp^*}\left(|u|^p+|v|^p\right)^{\frac{p^*}{p}}dx\biggr)^{\frac{p}{p^*}\cdot\frac{p^*}{p}}\\ &\le K\biggl(\biggl(\int_\Omega |x|^{-bp^*}|u|^{p^*}dx\biggr)^{\frac{p}{p^*}} + \biggl(\int_\Omega |x|^{-bp^*}|v|^{p^*}dx\biggr)^{\frac{p}{p^*}}\biggr)^{\frac{p^*}{p}}\\ &\le KC^{\frac{p^*}{p}}\left(\|u\|^p+\|v\|^p\right)^{\frac{p^*}{p}} = KC^{\frac{p^*}{p}}\left\|(u,v)\right\|^{p^*}. \qquad (2.8)\end{aligned}$$

Also, by the Hölder and Caffarelli–Kohn–Nirenberg's inequalities, we have

$$\begin{aligned}&\frac{p^*-p}{p^*-q}\left\|(u,v)\right\|^p = K_{\lambda,\mu}(u,v)\\ &\quad= \int_\Omega \lambda f|x|^{-\beta}|u|^q\,dx + \int_\Omega \mu f|x|^{-\beta}|v|^q\,dx\\ &\quad\le |\lambda|\biggl(\int_\Omega \left(f|x|^{-\beta}\right)^{p_0}dx\biggr)^{1/p_0}\biggl(\int_\Omega |u|^{p_1}dx\biggr)^{q/p_1}\\ &\quad+ |\mu|\biggl(\int_\Omega \left(f|x|^{-\beta}\right)^{p_0}dx\biggr)^{1/p_0}\biggl(\int_\Omega |v|^{p_1}dx\biggr)^{q/p_1}\\ &\quad\le |\lambda|\|f\|_{L^{p_0}(\Omega,|x|^{-\beta})}\|u\|^q_{L^{p_1}} + |\mu|\|f\|_{L^{p_0}(\Omega,|x|^{-\beta})}\|v\|^q_{L^{p_1}}\\ &\quad\le C^{\frac{q}{p}}\left(|\lambda|\|f\|_{L^{p_0}(\Omega,|x|^{-\beta})}\|u\|^q + |\mu|\|f\|_{L^{p_0}(\Omega,|x|^{-\beta})}\|v\|^q\right)\\ &\quad\le C^{\frac{q}{p}}\Bigl(\left(|\lambda|\|f\|_{L^{p_0}(\Omega,|x|^{-\beta})}\right)^{\frac{p}{p-q}} + \left(|\mu|\|f\|_{L^{p_0}(\Omega,|x|^{-\beta})}\right)^{\frac{p}{p-q}}\Bigr)^{\frac{p-q}{p}}\left\|(u,v)\right\|^q.\end{aligned}$$

Thus

$$\left\|(u,v)\right\| \ge \left(\frac{p-q}{K(p^*-q)}C^{-\frac{p^*}{p}}\right)^{\frac{1}{p^*-p}},$$

and

$$\|(u,v)\| \le \left(\frac{p^*-q}{p^*-p}C^{\frac{q}{p}}\right)^{\frac{1}{p-q}} \left[\left(|\lambda|\|f\|_{L^{p_0}(\Omega,|x|^{-\beta})}\right)^{\frac{p}{p-q}} + \left(|\mu|\|f\|_{L^{p_0}(\Omega,|x|^{-\beta})}\right)^{\frac{p}{p-q}}\right]^{\frac{1}{p}}.$$

This implies

$$\left(|\lambda|\|f\|_{L^{p_0}(\Omega,|x|^{-\beta})}\right)^{\frac{p}{p-q}} + \left(|\mu|\|f\|_{L^{p_0}(\Omega,|x|^{-\beta})}\right)^{\frac{p}{p-q}} \ge \Lambda.$$

This is a contradiction! Therefore, we can conclude that there exists $\Lambda > 0$ such that for

$$0 < \left(|\lambda|\|f\|_{L^{p_0}(\Omega,|x|^{-\beta})}\right)^{\frac{p}{p-q}} + \left(|\mu|\|f\|_{L^{p_0}(\Omega,|x|^{-\beta})}\right)^{\frac{p}{p-q}} < \Lambda,$$

we have $N^0_{\lambda,\mu} = \emptyset$. □

Lemma 4. *The energy functional $J_{\lambda,\mu}$ is coercive and bounded below on $N_{\lambda,\mu}$.*

Proof. If $(u,v) \in N_{\lambda,\mu}$, then by the Hölder inequality and Caffarelli–Kohn–Nirenberg's inequality, we can get

$$\begin{aligned} J_{\lambda,\mu}(u,v) &= \frac{p^*-p}{pp^*}\|(u,v)\|^p - \frac{p^*-q}{qp^*}K_{\lambda,\mu}(u,v) \\ &\ge \frac{p^*-p}{pp^*}\|(u,v)\|^p - \frac{p^*-q}{qp^*}C^{\frac{q}{p}}\left[\left(|\lambda|\|f\|_{L^{p_0}(\Omega,|x|^{-\beta})}\right)^{\frac{p}{p-q}} \right. \\ &\quad \left. + \left(|\mu|\|f\|_{L^{p_0}(\Omega,|x|^{-\beta})}\right)^{\frac{p}{p-q}}\right]^{\frac{p-q}{p}} \|(u,v)\|^q. \end{aligned}$$

Since $1 < q < p$, we see that $J_{\lambda,\mu}$ is coercive and bounded below on $N_{\lambda,\mu}$. □

Furthermore, similar to the argument in Brown and Zhang [4, Theorem 2.3] (or see Binding [4], Drábek, and Huang [11]), we can conclude the following result.

Lemma 5. *Assume that (u_0,v_0) is a local minimizer for $J_{\lambda,\mu}$ on $N_{\lambda,\mu}$ and that $(u_0,v_0) \notin N^0_{\lambda,\mu}$. Then $J'_{\lambda,\mu}(u_0,v_0) = 0$ in W^{-1} (the dual space of Sobolev space W).*

By Lemma 3, we let

$$\begin{aligned} \Theta_{\Lambda_0} = \Big\{(\lambda,\mu) \in \mathbb{R}^2 \setminus \{(0,0)\} : 0 < \left(|\lambda|\|f\|_{L^{p_0}(\Omega,|x|^{-\beta})}\right)^{\frac{p}{p-q}} \\ + \left(|\mu|\|f\|_{L^{p_0}(\Omega,|x|^{-\beta})}\right)^{\frac{p}{p-q}} < \Lambda_0\Big\}, \end{aligned}$$

where $\Lambda_0 = \left(\frac{q}{p}\right)^{\frac{p}{p-q}}\Lambda < \Lambda$. If $(\lambda,\mu) \in \Theta_{\Lambda_0}$, we have $N_{\lambda,\mu} = N^+_{\lambda,\mu} \cup N^-_{\lambda,\mu}$. Define

$$\theta_{\lambda,\mu} = \inf_{(u,v)\in N_{\lambda,\mu}} J_{\lambda,\mu}(u,v), \quad \theta^+_{\lambda,\mu} = \inf_{(u,v)\in N^+_{\lambda,\mu}} J_{\lambda,\mu}(u,v),$$
$$\theta^-_{\lambda,\mu} = \inf_{(u,v)\in N^-_{\lambda,\mu}} J_{\lambda,\mu}(u,v).$$

Lemma 6. *There exists a positive number Λ_0 such that if $(\lambda,\mu)\in\Theta_{\Lambda_0}$, then*

(i) $\theta_{\lambda,\mu}<\theta^+_{\lambda,\mu}<0$;

(ii) *there exists $d_0=d_0(p,q,N,K,C,\lambda,\mu)>0$ such that $\theta^-_{\lambda,\mu}>d_0$.*

Proof. (i) For $(u,v)\in N^+_{\lambda,\mu}$, by (2.6), we have

$$K_{\lambda,\mu}(u,v)\ge\frac{p^*-p}{p^*-q}\|(u,v)\|^p$$

and so

$$\begin{aligned}J_{\lambda,\mu}(u,v)&=\left(\frac{1}{p}-\frac{1}{p^*}\right)\|(u,v)\|^p-\left(\frac{1}{q}-\frac{1}{p^*}\right)K_{\lambda,\mu}(u,v)\\&\le\left(\frac{1}{p}-\frac{1}{p^*}\right)\|(u,v)\|^p-\left(\frac{1}{q}-\frac{1}{p^*}\right)\frac{p^*-p}{p^*-q}\|(u,v)\|^p\\&\le\frac{p^*-p}{p^*}\left(\frac{1}{p}-\frac{1}{q}\right)\|(u,v)\|^p<0.\end{aligned}$$

Thus, from definition of $\theta_{\lambda,\mu}$ and $\theta^+_{\lambda,\mu}$, we can deduce that $\theta_{\lambda,\mu}<\theta^+_{\lambda,\mu}<0$.

(ii) For $(u,v)\in N^-_{\lambda,\mu}$, by Lemma 3,

$$\|(u,v)\|\ge\left(\frac{p-q}{K(p^*-q)}\right)^{\frac{1}{p^*-p}}C^{-\frac{p^*}{p(p^*-p)}}.$$

Moreover, by Lemma 4,

$$\begin{aligned}J_{\lambda,\mu}(u,v)&\ge\frac{p^*-p}{pp^*}\|(u,v)\|^p-\frac{p^*-q}{qp^*}C^{\frac{q}{p}}\Big[\big(|\lambda|\|f\|_{L^{p_0}(\Omega,|x|^{-\beta})}\big)^{\frac{p}{p-q}}\\&\quad+\big(|\mu|\|f\|_{L^{p_0}(\Omega,|x|^{-\beta})}\big)^{\frac{p}{p-q}}\Big]^{\frac{p-q}{p}}\|(u,v)\|^q\\&=\|(u,v)\|^q\bigg[\frac{p^*-p}{pp^*}\|(u,v)\|^{p-q}-\frac{p^*-q}{qp^*}C^{\frac{q}{p}}\Big(\big(|\lambda|\|f\|_{L^{p_0}(\Omega,|x|^{-\beta})}\big)^{\frac{p}{p-q}}\\&\quad+\big(|\mu|\|f\|_{L^{p_0}(\Omega,|x|^{-\beta})}\big)^{\frac{p}{p-q}}\Big)^{\frac{p-q}{p}}\bigg]\\&\ge\left(\frac{p-q}{K(p^*-q)}\right)^{\frac{q}{p^*-p}}C^{-\frac{qp^*}{p(p^*-p)}}\bigg[\frac{p^*-p}{pp^*}\|(u,v)\|^{p-q}-\frac{p^*-q}{qp^*}C^{\frac{q}{p}}\\&\quad\times\Big(\big(|\lambda|\|f\|_{L^{p_0}(\Omega,|x|^{-\beta})}\big)^{\frac{p}{p-q}}+\big(|\mu|\|f\|_{L^{p_0}(\Omega,|x|^{-\beta})}\big)^{\frac{p}{p-q}}\Big)^{\frac{p-q}{p}}\bigg].\end{aligned}$$

Thus, if

$$0<\big(|\lambda|\|f\|_{L^{p_0}(\Omega,|x|^{-\beta})}\big)^{\frac{p}{p-q}}+\big(|\mu|\|f\|_{L^{p_0}(\Omega,|x|^{-\beta})}\big)^{\frac{p}{p-q}}<\Lambda_0,$$

then for each $(u,v)\in N^-_{\lambda,\mu}$ we have

$$J_{\lambda,\mu}(u,v)\ge d_0=d_0(p,q,N,K,C,\lambda,\mu)>0.\qquad\square$$

For each $(u,v) \in W \setminus \{(0,0)\}$ such that $\int_\Omega F(x,u^+,v^+)\,dx > 0$, let

$$t_{\max} = \left(\frac{(p-q)\|(u,v)\|^p}{(p^*-q)\int_\Omega |x|^{-bp^*} F(x,u,v)\,dx} \right)^{\frac{1}{p^*-p}}.$$

Lemma 7. *Assume that*

$$0 < \left(|\lambda| \|f\|_{L^{p_0}(\Omega,|x|^{-\beta})}\right)^{\frac{p}{p-q}} + \left(|\mu| \|f\|_{L^{p_0}(\Omega,|x|^{-\beta})}\right)^{\frac{p}{p-q}} < \Lambda_0.$$

Then, for every $(u,v) \in W$ with $\int_\Omega F(x,u^+,v^+)\,dx > 0$ there exists $t_{\max} > 0$ such that

(i) *if $K_{\lambda,\mu}(u,v) \le 0$, then, there is a unique $t^- > t_{\max}$ such that $(t^-u, t^-v) \in N^-_{\lambda,\mu}$ and*

$$J_{\lambda,\mu}(t^-u,t^-v) = \sup_{t\ge 0} J_{\lambda,\mu}(tu,tv);$$

(ii) *if $K_{\lambda,\mu}(u,v) > 0$, then, there are unique t^+ and t^- with $0 < t^+ < t_{\max} < t^-$ such that $(t^\pm u, t^\pm v) \in N^\pm_{\lambda,\mu}$ and*

$$J_{\lambda,\mu}(t^+u,t^+v) = \inf_{0\le t\le t_{\max}} J_{\lambda,\mu}(tu,tv), \quad J_{\lambda,\mu}(t^-u,t^-v) = \sup_{t\ge 0} J_{\lambda,\mu}(tu,tv).$$

Proof. Fix $(u,v) \in W$ with $\int_\Omega F(x,u,v)\,dx > 0$, let

$$m(t) = t^{p-q}\big\|(u,v)\big\|^p - t^{p^*-q}\int_\Omega |x|^{-bp^*} F(x,u,v)\,dx,$$

for $t \ge 0$. Clearly, $m(0) = 0$ and $m(t) \to -\infty$ as $t \to \infty$. Since

$$m'(t) = (p-q)t^{p-q-1}\big\|(u,v)\big\|^p - (p^*-q)t^{p^*-q-1}\int_\Omega F(x,u,v)\,dx,$$

there is a unique $t_{\max} > 0$ such that $m(t)$ achieves its maximum at $t_{\max}$, increasing for $t \in [0,t_{\max})$ and decreasing for $t \in (t_{\max},\infty)$ with $\lim_{t\to\infty} m(t) = -\infty$. Clearly, $(tu,tv) \in N^+_{\lambda,\mu}$ (or $N^-_{\lambda,\mu}$) if and only if $m'(t) > 0$ (or < 0). Moreover,

$$\begin{aligned}
m(t_{\max}) &= \left(\frac{(p-q)\|(u,v)\|^p}{(p^*-q)\int_\Omega |x|^{-bp^*} F(x,u,v)\,dx} \right)^{\frac{p-q}{p^*-p}} \big\|(u,v)\big\|^p \\
&\quad - \left(\frac{(p-q)\|(u,v)\|^p}{(p^*-q)\int_\Omega |x|^{-bp^*} F(x,u,v)\,dx} \right)^{\frac{p^*-q}{p^*-p}} \int_\Omega |x|^{-bp^*} F(x,u,v)\,dx \\
&= \big\|(u,v)\big\|^q \left[\left(\frac{p-q}{p^*-q}\right)^{\frac{p-q}{p^*-p}} - \left(\frac{p-q}{p^*-q}\right)^{\frac{p^*-q}{p^*-p}} \right] \\
&\quad \times \left(\frac{\|(u,v)\|^{p^*}}{\int_\Omega |x|^{-bp^*} F(x,u,v)\,dx} \right)^{\frac{p-q}{p^*-p}} \\
&\ge \left(\frac{p-q}{p^*-q}\right)^{\frac{p-q}{p^*-p}} \left(\frac{p^*-p}{p^*-q}\right) \left(\frac{1}{KC^{\frac{p^*}{p}}}\right)^{\frac{p-q}{p^*-p}} \big\|(u,v)\big\|^q.
\end{aligned}$$

(i) $K_{\lambda,\mu}(u,v) \leq 0$, then, there is unique $t^- > t_{\max}$ such that $m(t^-) = K_{\lambda,\mu}(u,v)$ and $m'(t^-) < 0$. Now,

$$(p-q)(t^-)^p \big\|(u,v)\big\|^p - (p^*-q)(t^-)^{p^*} \int_\Omega |x|^{-bp^*} F(x,u,v)\,dx \\ = (t^-)^{q+1} m(t^-) < 0,$$

and

$$\big\langle J'_{\lambda,\mu}(t^-u,t^-v),(t^-u,t^-v)\big\rangle = (t^-)^q\big[m(t^-) - K_{\lambda,\mu}(u,v)\big] = 0.$$

Thus, $(t^-u,t^-v) \in N^-_{\lambda,\mu}$. Since for $t > t_{\max}$, we have $m'(t) < 0$ and $m''(t) < 0$. Subsequently,

$$J_{\lambda,\mu}(t^-u,t^-v) = \sup_{t\geq 0} J_{\lambda,\mu}(tu,tv).$$

(ii) $K_{\lambda,\mu}(u,v) > 0$. For

$$0 < \big(|\lambda|\|f\|_{L^{p_0}(\Omega,|x|^{-\beta})}\big)^{\frac{p}{p-q}} + \big(|\mu|\|f\|_{L^{p_0}(\Omega,|x|^{-\beta})}\big)^{\frac{p}{p-q}} < \Lambda_0 < \Lambda,$$

we have

$$\begin{aligned} m(0) &= 0 < K_{\lambda,\mu}(u,v) \\ &\leq C^{\frac{q}{p}}\Big(\big(|\lambda|\|f\|_{L^{p_0}(\Omega,|x|^{-\beta})}\big)^{\frac{p}{p-q}} + \big(|\mu|\|f\|_{L^{p_0}(\Omega,|x|^{-\beta})}\big)^{\frac{p}{p-q}}\Big)^{\frac{p-q}{p}} \big\|(u,v)\big\|^q \\ &\leq \left(\frac{p-q}{p^*-q}\right)^{\frac{p-q}{p^*-p}} \left(\frac{p^*-p}{p^*-q}\right)\left(\frac{1}{KC^{\frac{p^*}{p}}}\right)^{\frac{p-q}{p^*-p}} \big\|(u,v)\big\|^q \leq m(t_{\max}), \end{aligned}$$

there are unique t^+ and t^- such that $0 < t^+ < t_{\max} < t^-$,

$$m(t^+) = K_{\lambda,\mu}(u,v) = m(t^-), \quad m'(t^+) > 0 > m'(t^-).$$

We have $(t^+u,t^+v) \in N^+_{\lambda,\mu}$, $(t^-u,t^-v) \in N^-_{\lambda,\mu}$; and

$$J_{\lambda,\mu}(t^-u,t^-v) \geq J_{\lambda,\mu}(tu,tv) \geq J_{\lambda,\mu}(t^+u,t^+v), \quad \forall t \in [t^+,t^-],$$
$$J_{\lambda,\mu}(t^+u,t^+v) \leq J_{\lambda,\mu}(tu,tv), \quad \forall t \in [0,t_{\max}].$$

Thus

$$J_{\lambda,\mu}(t^+u,t^+v) = \inf_{0\leq t\leq t_{\max}} J_{\lambda,\mu}(tu,tv), \quad J_{\lambda,\mu}(t^-u,t^-v) = \sup_{t\geq 0} J_{\lambda,\mu}(tu,tv).$$

This completes the proof. □

3 Existence of solutions

Now, we can state our main results.

Theorem 1. *Suppose R_0 and c_0 are positive constants with $B(0,3R_0) \subset \Omega$. In addition to* (H1)–(H3) *and* (A1) *hold. Then, there exists $\Lambda > 0$ such that problem* (1.1) *has a positive solution for each $f \in L^{p_0}(\Omega, |x|^{-\beta})$ satisfying $f(x) \geq 0$ for a.e. $x \in B(0,3R_0)$,*

$$\left(1 + R^{\frac{p(1+a-b)(N-p-ap)}{(p-1)(N-p(1+a-b))}}\right)^{-\frac{(N-p(1+a-b))q}{p(1+a-b)}} R^{N-\beta} \inf_{B(0,2R)} f \geq c_0, \text{ for some } R \in (0, R_0],$$

and the parameters λ, μ satisfy

$$0 < \left(|\lambda| \|f\|_{L^{p_0}(\Omega,|x|^{-\beta})}\right)^{\frac{p}{p-q}} + \left(|\mu| \|g\|_{L^{p_0}(\Omega,|x|^{-\beta})}\right)^{\frac{p}{p-q}} < \Lambda.$$

Theorem 2. *Suppose R_0 and c_0 are positive constants with $B(0,3R_0) \subset \Omega$. In addition to* (H1)–(H3) *and* (A1) *hold. Then, there exists $\Lambda_0 > 0$ such that problem* (1.1) *has at least two positive solutions (u_0^+, v_0^+) and (u_0^-, v_0^-) for each $f \in L^{p_0}(\Omega, |x|^{-\beta})$ satisfying $f(x) \geq 0$ for a.e. $x \in B(0,3R_0)$,*

$$\left(1+R^{\frac{p(1+a-b)(N-p-ap)}{(p-1)(N-p(1+a-b))}}\right)^{-\frac{(N-p(1+a-b))q}{p(1+a-b)}} R^{N-\beta} \inf_{B(0,2R)} f \geq c_0, \text{ for some } R \in (0, R_0],$$

and the parameters λ, μ satisfy

$$0 < \left(|\lambda| \|f\|_{L^{p_0}(\Omega,|x|^{-\beta})}\right)^{\frac{p}{p-q}} + \left(|\mu| \|f\|_{L^{p_0}(\Omega,|x|^{-\beta})}\right)^{\frac{p}{p-q}} < \Lambda_0.$$

Before given the proofs of Theorems 1 and 2, we need the following lemma.

Lemma 8. (i) *If $0 < (|\lambda| \|f\|_{L^{p_0}(\Omega,|x|^{-\beta})})^{\frac{p}{p-q}} + (|\mu| \|\boldsymbol{f}\|_{L^{p_0}(\Omega,|x|^{-\beta})})^{\frac{p}{p-q}} < \Lambda$, then there exists a $(PS)_{\theta_{\lambda,\mu}}$-sequence $\{(u_n, v_n)\} \subset N_{\lambda,\mu}$ in W for $J_{\lambda,\mu}$.*

(ii) *If $0 < (|\lambda| \|f\|_{L^{p_0}(\Omega,|x|^{-\beta})})^{\frac{p}{p-q}} + (|\mu| \|f\|_{L^{p_0}(\Omega,|x|^{-\beta})})^{\frac{p}{p-q}} < \Lambda_0$, then there exists a $(PS)_{\theta^-_{\lambda,\mu}}$-sequence $\{(u_n, v_n)\} \subset N^-_{\lambda,\mu}$ in W for $J_{\lambda,\mu}$.*

Proof. The proof is almost the same as that in Wu [21]. □

Theorem 3. *If $0 < (|\lambda| \|f\|_{L^{p_0}(\Omega,|x|^{-\beta})})^{\frac{p}{p-q}} + (|\mu| \|f\|_{L^{p_0}(\Omega,|x|^{-\beta})})^{\frac{p}{p-q}} < \Lambda$ and* (H1)–(H3) *hold, then $J_{\lambda,\mu}$ has a minimizer (u_0^+, v_0^+) in $N^+_{\lambda,\mu}$ and it satisfies*

(i) $J_{\lambda,\mu}(u_0^+, v_0^+) = \theta^+_{\lambda,\mu}$,

(ii) (u_0^+, v_0^+) *is a positive solution of* (1.1).

Proof. By the Lemma 8(i), there exist a minimizing sequence $\{(u_n, v_n)\}$ for $J_{\lambda,\mu}$ on $N_{\lambda,\mu}$ such that

$$J_{\lambda,\mu}(u_n, v_n) = \theta_{\lambda,\mu} + o(1) \quad \text{and} \quad J'_{\lambda,\mu}(u_n, v_n) = o(1) \quad \text{in } W^{-1}. \tag{3.1}$$

Then, by Lemma 4 and the continuity of the embedding theorem, there is a subsequence $\{(u_n, v_n)\}$ and $(u_0^+, v_0^+) \in W$ such that

$$\begin{cases} u_n \rightharpoonup u_0^+, & v_n \rightharpoonup v_0^+, \quad \text{weakly in } W_0^{1,p}(\Omega, |x|^{-ap}), \\ u_n \to u_0^+, & v_n \to v_0^+, \quad \text{strongly in } L^q(\Omega, |x|^{-\beta}), \\ u_n \to u_0^+, & v_n \to v_0^+, \quad \text{a.e in } \Omega, \end{cases} \tag{3.2}$$

as $n \to \infty$. This implies that

$$K_{\lambda,\mu}(u_n, v_n) \to K_{\lambda,\mu}(u_0^+, v_0^+), \quad \text{as } n \to \infty.$$

By (3.1) and (3.2), it is easy to prove that (u_0^+, v_0^+) is a weak solution of problem (1.1). Since

$$J_{\lambda,\mu}(u_n, v_n) = \frac{p^*-p}{pp^*}\|(u_n, v_n)\|^p - \frac{p^*-q}{qp^*}K_{\lambda,\mu}(u_n, v_n) \geq -\frac{p^*-q}{qp^*}K_{\lambda,\mu}(u_n, v_n),$$

and by Lemma 4(i),

$$J_{\lambda,\mu}(u_n, v_n) \to \theta_{\lambda,\mu} < 0 \quad \text{as } n \to \infty.$$

Letting $n \to \infty$, we see that $K_{\lambda,\mu}(u_0^+, v_0^+) > 0$.

Now, we prove that

$$\begin{cases} u_n \to u_0^+, & \text{strongly in } W_0^{1,p}(\Omega, |x|^{-ap}), \\ v_n \to v_0^+, & \text{strongly in } W_0^{1,p}(\Omega, |x|^{-ap}), \end{cases}$$

and $J_{\lambda,\mu}(u_0^+, v_0^+) = \theta_{\lambda,\mu}$.

By applying Fatou's lemma and $(u_0^+, v_0^+) \in N_{\lambda,\mu}$, we get

$$\begin{aligned} \theta_{\lambda,\mu} \leq J_{\lambda,\mu}(u_0^+, v_0^+) &= \left(\frac{1}{p} - \frac{1}{p^*}\right)\|(u_0^+, v_0^+)\|^p - \frac{p^*-q}{qp^*}K_{\lambda,\mu}(u_0^+, v_0^+) \\ &\leq \liminf_{n\to\infty}\left[\left(\frac{1}{p} - \frac{1}{p^*}\right)\|(u_n, v_n)\|^p - \frac{p^*-q}{qp^*}K_{\lambda,\mu}(u_n, v_n)\right] \\ &\leq \liminf_{m\to\infty} J_{\lambda,\mu}(u_n, v_n) = \theta_{\lambda,\mu}. \end{aligned}$$

This implies that

$$J_{\lambda,\mu}(u_0^+, v_0^+) = \theta_{\lambda,\mu}, \qquad \lim_{n\to\infty}\|(u_n, v_n)\|^p = \|(u_0^+, v_0^+)\|^p.$$

Then, $u_n \to u_0^+$ strongly in $W_0^{1,p}(\Omega, |x|^{-ap})$ and $v_n \to v_0^+$ strongly in $W_0^{1,p}(\Omega, |x|^{-ap})$.

Moreover, we have $(u_0^+, v_0^+) \in N_{\lambda,\mu}^+$. In fact, if $(u_0^+, v_0^+) \in N_{\lambda,\mu}^-$, by Lemma 7, there are unique t_0^+ and t_0^- such that $(t_0^+ u_0^+, t_0^+ v_0^+) \in N_{\lambda,\mu}^+$, $(t_0^- u_0^+, t_0^- v_0^+) \in N_{\lambda,\mu}^-$ and $t_0^+ < t_0^- = 1$. Since

$$\frac{d}{dt}J_{\lambda,\mu}(t_0^+ u_0^+, t_0^+ v_0^+) = 0 \quad \text{and} \quad \frac{d^2}{dt^2}J_{\lambda,\mu}(t_0^+ u_0^+, t_0^+ v_0^+) > 0,$$

there exist $t_0^+ < \bar{t} \leq t_0^-$ such that $J_{\lambda,\mu}(t_0^+ u^+, t_0^+ v_0^+) < J_{\lambda,\mu}(\bar{t}_0 u_0^+, \bar{t}_0 v_0^+)$. By Lemma 7, we have

$$J_{\lambda,\mu}(t_0^+ u_0^+, t_0^+ u_0^+) < J_{\lambda,\mu}(\bar{t}_0 u_0^+, \bar{t}_0 u_0^+) \leq J_{\lambda,\mu}(t_0^- u_0^+, t_0^- v_0^+) = J_{\lambda,\mu}(u_0^+, v_0^+)$$

which contradicts $J_{\lambda,\mu}(u_0^+, v_0^+) = \theta_{\lambda,\mu}^+$. Since $J_{\lambda,\mu}(u_0^+, v_0^+) = J_{\lambda,\mu}(|u_0^+|, |v_0^+|)$ and $(|u_0^+|, |v_0^+|) \in N_{\lambda,\mu}^+$. By Lemma 5, we may assume that (u_0^+, v_0^+) is a non-negative solution of problem (1.1). By the maximum principle, it follows that $u_0^+ > 0$, $v_0^+ > 0$ in Ω. □

The following two lemmas are similar to that in Hsu [16].

Lemma 9. *If $\{(u_n,v_n)\} \in W$ is a $(PS)_c$-sequence for $J_{\lambda,\mu}$ with $(u_n,v_n) \rightharpoonup (u,v)$ in W, then $J'_{\lambda,\mu}(u,v)=0$, and there exists a positive constant Υ depending on p, q, N and C, such that $J_{\lambda,\mu}(u,v) \geq -\Upsilon((|\lambda|\|f\|_{L^{p_0}(\Omega,|x|^{-\beta})})^{\frac{p}{p-q}} + (|\mu|\|f\|_{L^{p_0}(\Omega,|x|^{-\beta})})^{\frac{p}{p-q}})$.*

Lemma 10. *If $\{(u_n,v_n)\} \in W$ be a $(PS)_c$-sequence for $J_{\lambda,\mu}$, then $\{(u_n,v_n)\}$ is bounded in W.*

Denote

$$A_F = \inf_{(u,v)\in W}\left\{\frac{\|(u,v)\|^p}{(\int_\Omega |x|^{-bp^*}F(x,u,v)\,dx)^{\frac{p}{p^*}}}\right\}.$$

Now, we need the following proposition.

Proposition 1. [13] *Suppose that $F \in C^1(\Omega\times\mathbb{R}\times\mathbb{R},\mathbb{R}^+)$ is positively homogeneous of degree p^* with $p^*>1$. Then, there exists $M_F>0$ such that*

$$\big|F(x,u,v)\big| \leq M_F\big(|u|^{p^*}+|v|^{p^*}\big), \quad \forall (x,u,v)\in\Omega\times\mathbb{R}\times\mathbb{R},$$

where $M_F=\max\{F(x,u,v) \mid x\in\Omega,\ u,v\in\mathbb{R},\ |u|^{p^*}+|v|^{p^*}=1\}$.

Also, we need the following version of Brèzis–Lieb lemma [5].

Lemma 11. *Consider $F\in C^1(\overline{\Omega},(\mathbb{R}^+)^2)$ with $F(x,0,0)=0$ and*

$$\big|F_u(x,u,v)\big|,\big|F_v(x,u,v)\big| \leq C_1\big(|u|^{p^*-1}+|v|^{p^*-1}\big)$$

for some $1\leq p^*<\infty$, $C_1>0$. Let (u_n,v_n) be bounded sequence in $L^{p^*}(\overline{\Omega},|x|^{-bp^*})$, and such that $(u_n,v_n)\rightharpoonup(u,v)$ weakly in W. Then one has

$$\int_\Omega |x|^{-bp^*}F(x,u_n,v_n)\,dx \to \int_\Omega |x|^{-bp^*}F(x,u_n-u,v_n-v)\,dx + \int_\Omega |x|^{-bp^*}F(x,u,v)\,dx \quad \text{as } n\to\infty.$$

Proof. We will follow the approach presented in [5, 13] to give the proof of this lemma. Using the mean value theorem, for given $0<|\theta|<1$, it follows that

$$\begin{aligned}
&|x|^{-bp^*}\big|F(x,u_n,v_n)-F(x,u_n-u,v_n-v)\big| \\
&\quad= \big||x|^{-bp^*}\nabla F(x,u_n-u+\theta u,v_n-v+\theta v)\cdot(u,v)\big| \\
&\quad\leq C_1|x|^{-bp^*}\big(|u_n-u+\theta u|^{p^*-1}+|v_n-v+\theta v|^{p^*-1}\big)|u| \\
&\qquad C_1|x|^{-bp^*}\big(|u_n-u+\theta u|^{p^*-1}+|v_n-v+\theta v|^{p^*-1}\big)|v| \\
&\quad\leq C|x|^{-bp^*}\big[|u_n-u|^{p^*-1}|u|+|u|^{p^*}+|v_n-v|^{p^*-1}|u|+|v|^{p^*-1}|u| \\
&\qquad |u_n-u|^{p^*-1}|v|+|u|^{p^*-1}|v|+|v_n-v|^{p^*-1}|v|+|v|^{p^*}\big] \\
&\quad\leq C|x|^{-bp^*}\big[|u_n-u|^{p^*-1}|u|+|v_n-v|^{p^*-1}|v|+|u_n-u|^{p^*-1}|v| \\
&\qquad |v_n-v|^{p^*-1}|u|+|u|^{p^*}+|v|^{p^*}+|u|^{p^*-1}|v|+|v|^{p^*-1}|u|\big].
\end{aligned}$$

Hence, for any $\epsilon > 0$, applying the Young inequality to the last inequality, there exists $C_\epsilon > 0$ such that

$$\begin{aligned} &|x|^{-bp^*}\bigl|F(x,u_n,v_n)-F(x,u_n-u,v_n-v)\bigr| \\ &\quad \le \epsilon|x|^{-bp^*}\bigl[|u_n-u|^{p^*}+|v_n-v|^{p^*}\bigr]+C_\epsilon|x|^{-bp^*}\bigl(|u|^{p^*}+|v|^{p^*}\bigr). \end{aligned}$$

Now we define the functions

$$\begin{aligned} f_n &= |x|^{-bp^*}\bigl|F(x,u_n,v_n)-F(x,u_n-u,v_n-v)-F(x,u,v)\bigr|, \\ g_n &= f_n-\epsilon|x|^{-bp^*}\bigl(|u_n-u|^{p^*}+|v_n-v|^{p^*}\bigr). \end{aligned}$$

Then

$$\begin{aligned} f_n &\le \epsilon|x|^{-bp^*}\bigl(|u_n-u|^{p^*}+|v_n-v|^{p^*}\bigr)+C_\epsilon|x|^{-bp^*}\bigl(|u|^{p^*}+|v|^{p^*}\bigr) \\ &\qquad +|x|^{-bp^*}\bigl|F(x,u,v)\bigr|, \\ g_n &\le |x|^{-bp^*}\bigl|F(x,u,v)\bigr|+C_\epsilon|x|^{-bp^*}\bigl(|u|^{p^*}+|v|^{p^*}\bigr) \\ &\le M_F|x|^{-bp^*}\bigl(|u|^{p^*}+|v|^{p^*}\bigr)+C_\epsilon|x|^{-bp^*}\bigl(|u|^{p^*}+|v|^{p^*}\bigr) \\ &\le (M_F+C_\epsilon)|x|^{-bp^*}\bigl(|u|^{p^*}+|v|^{p^*}\bigr)\in L^1\bigl(\Omega,|x|^{-bp^*}\bigr). \end{aligned}$$

Since $(u_n,v_n)\rightharpoonup(u,v)$ in W, we can assume that $u_n\to u$, $v_n\to v$ a.e. in Ω. Thus, $g_n\to 0$ a.e. in Ω as $n\to\infty$. The Lebesgue dominated convergence theorem implies that

$$\lim_{n\to\infty}\int_\Omega g_n(x)\,dx=0.$$

Therefore, we obtain

$$\begin{aligned} \limsup_{n\to\infty}\int_\Omega f_n(x)\,dx &\le \limsup_{n\to\infty}\int_\Omega \bigl(g_n(x)+\epsilon|x|^{-bp^*}\bigl(|u_n-u|^{p^*}+|v_n-v|^{p^*}\bigr)\bigr)\,dx \\ &\le \limsup_{n\to\infty}\int_\Omega g_n(x)\,dx+\epsilon\limsup_{n\to\infty}\int_\Omega |x|^{-bp^*}\bigl(|u_n-u|^{p^*}+|v_n-v|^{p^*}\bigr)\,dx \\ &\le C_\epsilon. \end{aligned}$$

By the arbitrariness of $\epsilon>0$, one has

$$\lim_{n\to\infty}\int_\Omega f_n(x)\,dx=0.$$

This completes the proof. □

Lemma 12. *$J_{\lambda,\mu}$ satisfies the $(PS)_{c_F}$ condition with c_F satisfying*

$$\begin{aligned} -\infty<c_F<c_\infty &= \Bigl(\frac{1}{p}-\frac{1}{p^*}\Bigr)A_F^{\frac{p^*}{p^*-p}} \\ &\qquad -\Upsilon\Bigl(\bigl(|\lambda|\|f\|_{L^{p_0}(\Omega,|x|^{-\beta})}\bigr)^{\frac{p}{p-q}}+\bigl(|\mu|\|f\|_{L^{p_0}(\Omega,|x|^{-\beta})}\bigr)^{\frac{p}{p-q}}\Bigr). \end{aligned}$$

Proof. Let $\{(u_n,v_n)\}\in W$ be a $(PS)_{c_F}$-sequence for $J_{\lambda,\mu}$ with $c_F\in(-\infty,c_\infty)$. It follows from Lemma 10 that $\{(u_n,v_n)\}$ is bounded in W, and then $(u_n,v_n)\rightharpoonup(u,v)$ up to a subsequence, (u,v) is a critical point of $J_{\lambda,\mu}$. Moreover, we may assume

$$\begin{cases} u_n\rightharpoonup u, & v_n\rightharpoonup v, \quad \text{weakly in } W_0^{1,p}(\Omega,|x|^{-ap}),\\ u_n\to u, & u_n\to u, \quad \text{strongly in } L^q(\Omega,|x|^{-\beta}),\\ u_n\to u, & u_n\to u, \quad \text{a.e. on } \Omega.\end{cases}$$

Hence, we have that $J'_{\lambda,\mu}(u,v)=0$ and

$$K_{\lambda,\mu}(u_n,v_n)\to K_{\lambda,\mu}(u,v),\quad \text{as } n\to\infty. \tag{3.3}$$

Let $\widetilde{u}_n=u_n-u$, $\widetilde{v}_n=v_n-v$. Then by Brèzis–Lieb lemma [5], we obtain

$$\|(\widetilde{u}_n,\widetilde{v}_n)\|^p\to\|(u_n,v_n)\|^p-\|(u,v)\|^p,\quad \text{as } n\to\infty, \tag{3.4}$$

and by Lemma 11,

$$\int_\Omega |x|^{-bp^*}F(x,\widetilde{u}_n,\widetilde{v}_n)\,dx\to\int_\Omega |x|^{-bp^*}F(x,u_n,v_n)\,dx - \int_\Omega |x|^{-bp^*}F(x,u,v)\,dx,\quad \text{as } n\to\infty. \tag{3.5}$$

Since $J_{\lambda,\mu}(u_n,v_n)=c_F+o(1)$, $J'_{\lambda,\mu}(u_n,v_n)=o(1)$ and (3.3)–(3.5), we can deduce that

$$\frac{1}{p}\|(\widetilde{u}_n,\widetilde{v}_n)\|^p-\frac{1}{p^*}\int_\Omega |x|^{-bp^*}F(x,\widetilde{u}_n,\widetilde{v}_n)\,dx=c_F-J_{\lambda,\mu}(u,v)+o(1), \tag{3.6}$$

$$\|(\widetilde{u}_n,\widetilde{v}_n)\|^p-\int_\Omega |x|^{-bp^*}F(x,\widetilde{u}_n,\widetilde{v}_n)\,dx=o(1).$$

Thus, we may assume that

$$\|(\widetilde{u}_n,\widetilde{v}_n)\|^p\to l,\quad \int_\Omega |x|^{-bp^*}F(x,\widetilde{u}_n,\widetilde{v}_n)\,dx\to l. \tag{3.7}$$

If $l=0$, the proof is completed. Assume $l>0$, then from (3.7), we obtain

$$A_F l^{\frac{p}{p^*}}=A_F\lim_{n\to\infty}\left(\int_\Omega |x|^{-bp^*}F(x,\widetilde{u}_n,\widetilde{v}_n)\,dx\right)^{p/p^*}\le\lim_{n\to\infty}\|(\widetilde{u}_n,\widetilde{v}_n)\|^p=l,$$

which implies that $l\ge A_F^{\frac{p^*}{p^*-p}}$.

In additional, from Lemma 9, (3.6) and (3.7), we get

$$c_F=\left(\frac{1}{p}-\frac{1}{p^*}\right)l+J_{\lambda,\mu}(u,v)\ge\left(\frac{1}{p}-\frac{1}{p^*}\right)A_F^{\frac{p^*}{p^*-p}} -\Upsilon\Big(\big(|\lambda|\|f\|_{L^{p_0}(\Omega,|x|^{-\beta})}\big)^{\frac{p}{p-q}}+\big(|\mu|\|f\|_{L^{p_0}(\Omega,|x|^{-\beta})}\big)^{\frac{p}{p-q}}\Big),$$

which contradicts

$$c_F < \left(\frac{1}{p} - \frac{1}{p^*}\right) A_F^{\frac{p^*}{p^*-p}} - \Upsilon\Big(\big(|\lambda| \|f\|_{L^{p_0}(\Omega,|x|^{-\beta})}\big)^{\frac{p}{p-q}} + \big(|\mu| \|f\|_{L^{p_0}(\Omega,|x|^{-\beta})}\big)^{\frac{p}{p-q}}\Big).$$

□

Lemma 13. *There exist a non-negative function* $(u,v) \in W \setminus \{(0,0)\}$ *and* $C^* > 0$ *such that for*

$$0 < (|\lambda| \|f\|_{L^{p_0}(\Omega,|x|^{-\beta})})^{\frac{p}{p-q}} + (|\mu| \|f\|_{L^{p_0}(\Omega,|x|^{-\beta})})^{\frac{p}{p-q}} < C^*,$$

we have

$$\sup_{t\geq 0} J_{\lambda,\mu}(tu,tv) < A_F.$$

In particular $\theta^-_{\lambda,\mu} < c_\infty$ *for all*

$$0 < \big(|\lambda| \|f\|_{L^{p_0}(\Omega,|x|^{-\beta})}\big)^{\frac{p}{p-q}} + \big(|\mu| \|f\|_{L^{p_0}(\Omega,|x|^{-\beta})}\big)^{\frac{p}{p-q}} < C^*.$$

Proof. We fix in Lemma 1 the constants $R_1 = R_0$ and $c_1 = c_0$. Now, we consider the functional $I : W \to \mathbb{R}$ defined by

$$I(u,v) = \frac{1}{p}\big\|(u,v)\big\|^p - \frac{1}{p^*}\int_\Omega |x|^{-bp^*} F(x,u,v)\, dx \quad \text{for all } (u,v) \in W.$$

Set $u_0 = e_1 u_\epsilon$, $v_0 = e_2 u_\epsilon$ and $(u_0,v_0) \in W$, where $(e_1,e_2) \in (\mathbb{R}^+)^2$, $e_1^p + e_2^p = 1$ and $\inf_{x\in\overline{\Omega}} F(x,e_1,e_2) \geq K$. Then by (H1) and (2.8), the definition of A_F and Lemma 1, we obtain that

$$\begin{aligned}
\sup_{t\geq 0} I(te_1u_\epsilon, te_2u_\epsilon) &\leq \left(\frac{1}{p} - \frac{1}{p^*}\right)\left(\frac{(e_1^p + e_2^p)\int_\Omega |x|^{-ap}|\triangle u_\epsilon|^p\, dx}{(\int_\Omega |x|^{-bp^*} F(x,e_1u_\epsilon,e_2u_\epsilon)\, dx)^{\frac{p}{p^*}}}\right)^{\frac{p^*}{p^*-p}} \\
&\leq \left(\frac{1}{p} - \frac{1}{p^*}\right)\left(\frac{\int_\Omega |x|^{-ap}|\triangle u_\epsilon|^p\, dx}{K^{\frac{p}{p^*}}(\int_\Omega |x|^{-bp^*}|u_\epsilon|^{p^*}\, dx)^{\frac{p}{p^*}}}\right)^{\frac{p^*}{p^*-p}} \\
&\leq \left(\frac{1}{p} - \frac{1}{p^*}\right)\left(\frac{1}{K^{\frac{p}{p^*}}}\right)^{\frac{p^*}{p^*-p}} \big(\widetilde{S}_{a,p,R} + O\big(\epsilon^{\frac{N-p(1+a-b)}{p(1+a-b)}}\big)\big)^{\frac{p^*}{p^*-p}} \\
&\leq \left(\frac{1}{p} - \frac{1}{p^*}\right)\left(\frac{1}{K^{\frac{p}{p^*}}}\right)^{\frac{p^*}{p^*-p}} \big(\widetilde{S}_{a,p,R} + O(\epsilon^{\frac{N-p(1+a-b)}{p(1+a-b)}})\big)^{\frac{p^*}{p^*-p}} \\
&\leq \left(\frac{1}{p} - \frac{1}{p^*}\right)\left(\frac{1}{K^{\frac{p}{p^*}}}\right)^{\frac{p^*}{p^*-p}} \big(\widetilde{S}_{a,p,R}^{\frac{p^*}{p^*-p}} + O(\epsilon^{\frac{N-p(1+a-b)}{p(1+a-b)}})\big) \\
&\leq \left(\frac{1}{p} - \frac{1}{p^*}\right) A_F^{\frac{p^*}{p^*-p}} + O\big(\epsilon^{\frac{N-p(1+a-b)}{p(1+a-b)}}\big),
\end{aligned} \tag{3.8}$$

where the following fact has been used:

$$\sup_{t\geq 0}\left(\frac{t^p}{p}A - \frac{t^{p^*}}{p^*}B\right) = \left(\frac{1}{p} - \frac{1}{p^*}\right)\left(\frac{A}{B^{\frac{p}{p^*}}}\right)^{\frac{p^*}{p^*-p}}, \quad A,B > 0.$$

We can choose $\delta_1 > 0$ such that for all

$$0 < \left(|\lambda| \|f\|_{L^{p_0}(\Omega,|x|^{-\beta})}\right)^{\frac{p}{p-q}} + \left(|\mu| \|f\|_{L^{p_0}(\Omega,|x|^{-\beta})}\right)^{\frac{p}{p-q}} < \delta_1,$$

we have

$$c_\infty = \frac{p^*-p}{pp^*} A_F^{\frac{p^*}{p^*-p}} - \Upsilon\left(\left(|\lambda| \|f\|_{L^{p_0}(\Omega,|x|^{-\beta})}\right)^{\frac{p}{p-q}} + \left(|\mu| \|f\|_{L^{p_0}(\Omega,|x|^{-\beta})}\right)^{\frac{p}{p-q}}\right) > 0.$$

Using the definitions of $J(u,v)$ and (u_0,v_0), we get

$$J_{\lambda,\mu}(tu_0,tv_0) \le \frac{t^p}{p} \left\|(u_0,u_0)\right\|^p \quad \text{for all } t \ge 0 \text{ and } \lambda,\mu > 0,$$

which implies that there exists $t_0 \in (0,1)$ satisfying

$$\sup_{0 \le t \le t_0} J_{\lambda,\mu}(t_0u_0,t_0u_0) < c_\infty,$$

for all

$$0 < \left(|\lambda| \|f\|_{L^{p_0}(\Omega,|x|^{-\beta})}\right)^{\frac{p}{p-q}} + \left(|\mu| \|f\|_{L^{p_0}(\Omega,|x|^{-\beta})}\right)^{\frac{p}{p-q}} < \delta_1.$$

Using the definitions of $J(u,v)$ and (u_0,v_0), and by (3.8), we have

$$\begin{aligned} \sup_{t \ge t_0} J_{\lambda,\mu}(t_0u_0,t_0u_0) &= \sup_{t \ge t_0}\left(I(t_0u_0,t_0u_0) - \frac{t^q}{q} K_{\lambda,\mu}(u_0^+,v_0^+)\right) \\ &\le \left(\frac{1}{p} - \frac{1}{p^*}\right) A_F^{\frac{p^*}{p^*-p}} + O\left(\epsilon^{\frac{N-p(1+a-b)}{p(1+a-b)}}\right) - \frac{t_0^q}{q} m(\lambda+\mu) \int\limits_{B(x_0,R_0)} |u_\epsilon|^q \, dx, \end{aligned} \tag{3.9}$$

where $m = \min\{e_1^q, e_2^q\}$. We observe that

$$\frac{(N-p(1+a-b))q}{p^2(1+a-b)} < \frac{N-p(1+a-b)}{p(1+a-b)}. \tag{3.10}$$

Suppose $q < \frac{(N-\beta)(p-1)}{N-p-ap}$. The inequalities (3.9), (3.10) and Lemma 1, imply

$$\begin{aligned} \sup_{t \ge t_0} J_{\lambda,\mu}(t_0u_0,t_0u_0) \le \left(\frac{1}{p} - \frac{1}{p^*}\right) A_F^{\frac{p^*}{p^*-p}} + O(\epsilon^{\frac{N-p(1+a-b)}{p(1+a-b)}}) \\ - \frac{t_0^q}{q} m(\lambda+\mu) O\left(\epsilon^{\frac{(N-p(1+a-b))q}{p^2(1+a-b)}}\right). \end{aligned} \tag{3.11}$$

Now, for all

$$\epsilon = \left(\left(|\lambda| \|f\|_{L^{p_0}(\Omega,|x|^{-\beta})}\right)^{\frac{p}{p-q}} + \left(|\mu| \|f\|_{L^{p_0}(\Omega,|x|^{-\beta})}\right)^{\frac{p}{p-q}}\right)^{\frac{p(1+a-b)}{N-p(1+a-b)}} \in (0,R_0),$$

we get

$$\begin{aligned} \sup_{t \ge t_0} J_{\lambda,\mu}(t_0u_0,t_0v_0) &\le \left(\frac{1}{p} - \frac{1}{p^*}\right) A_F^{\frac{p^*}{p^*-p}} + O\left(\left(|\lambda| \|f\|_{L^{p_0}(\Omega,|x|^{-\beta})}\right)^{\frac{p}{p-q}}\right. \\ &\quad \left. + \left(|\mu| \|f\|_{L^{p_0}(\Omega,|x|^{-\beta})}\right)^{\frac{p}{p-q}}\right) - \frac{t_0^q}{q} m(\lambda+\mu) \left(\left(|\lambda| \|f\|_{L^{p_0}(\Omega,|x|^{-\beta})}\right)^{\frac{p}{p-q}}\right. \\ &\quad \left. + \left(|\mu| \|f\|_{L^{p_0}(\Omega,|x|^{-\beta})}\right)^{\frac{p}{p-q}}\right)^{\frac{q}{p}}. \end{aligned}$$

Thus, we can choose $\delta_2 > 0$ such that for all $0 < (|\lambda| \|f\|_{L^{p_0}(\Omega,|x|^{-\beta})})^{\frac{p}{p-q}} + (|\mu| \|f\|_{L^{p_0}(\Omega,|x|^{-\beta})})^{\frac{p}{p-q}} < \delta_2$, we obtain

$$\begin{aligned} & O\big((|\lambda| \|f\|_{L^{p_0}(\Omega,|x|^{-\beta})})^{\frac{p}{p-q}} + (|\mu| \|f\|_{L^{p_0}(\Omega,|x|^{-\beta})})^{\frac{p}{p-q}} \big) \\ & \quad - \frac{t_0^q}{q} m(\lambda+\mu) \big((|\lambda| \|f\|_{L^{p_0}(\Omega,|x|^{-\beta})})^{\frac{p}{p-q}} + (|\mu| \|f\|_{L^{p_0}(\Omega,|x|^{-\beta})})^{\frac{p}{p-q}} \big)^{\frac{q}{p}} \\ & \quad \le -\Upsilon \big((|\lambda| \|f\|_{L^{p_0}(\Omega,|x|^{-\beta})})^{\frac{p}{p-q}} + (|\mu| \|f\|_{L^{p_0}(\Omega,|x|^{-\beta})})^{\frac{p}{p-q}} \big). \end{aligned}$$

If we set $C^* = \min\{\delta_1, R_0, \delta_2\}$ and

$$\epsilon = \big((|\lambda| \|f\|_{L^{p_0}(\Omega,|x|^{-\beta})})^{\frac{p}{p-q}} + (|\mu| \|f\|_{L^{p_0}(\Omega,|x|^{-\beta})})^{\frac{p}{p-q}} \big)^{\frac{p(1+a-b)}{N-p(1+a-b)}},$$

then for

$$0 < (|\lambda| \|f\|_{L^{p_0}(\Omega,|x|^{-\beta})})^{\frac{p}{p-q}} + (|\mu| \|f\|_{L^{p_0}(\Omega,|x|^{-\beta})})^{\frac{p}{p-q}} < C^*,$$

we have

$$\sup_{t\ge t_0} J_{\lambda,\mu}(t_0u_0, t_0u_0) \le c_\infty. \tag{3.12}$$

Similarly, let $q = \frac{(N-\beta)(p-1)}{N-p-ap}$, by inequalities (3.9), (3.10) and Lemma 1, one can get

$$\begin{aligned} \sup_{t\ge t_0} J_{\lambda,\mu}(t_0u_0, t_0u_0) \le & \left(\frac{1}{p} - \frac{1}{p^*}\right) A_F^{\frac{p^*}{p^*-p}} + O\big(\epsilon^{\frac{N-p(1+a-b)}{p(1+a-b)}}\big) \\ & - \frac{t_0^q}{q} m(\lambda+\mu) O\big(\epsilon^{\frac{(N-p(1+a-b))q}{p^2(1+a-b)}} |\ln\epsilon|\big). \end{aligned} \tag{3.13}$$

If $q > \frac{(N-\beta)(p-1)}{N-p-ap}$, then

$$\begin{aligned} \sup_{t\ge t_0} J_{\lambda,\mu}(t_0u_0, t_0u_0) \le & \left(\frac{1}{p} - \frac{1}{p^*}\right) A_F^{\frac{p^*}{p^*-p}} + O\big(\epsilon^{\frac{N-p(1+a-b)}{p(1+a-b)}}\big) \\ & - \frac{t_0^q}{q} m(\lambda+\mu) O\big(\epsilon^{\frac{(N-p(1+a-b))(p-1)[(N-\beta)p-(N-p-ap)q]}{p^2(1+a-b)(N-p-ap)}}\big), \end{aligned} \tag{3.14}$$

then, by (3.13) and (3.14), we have

$$\sup_{t\ge t_0} J_{\lambda,\mu}(t_0u_0, t_0u_0) \le c_\infty. \tag{3.15}$$

Finally, we prove that $\theta^-_{\lambda,\mu} < c_\infty$ for all

$$0 < (|\lambda| \|f\|_{L^{p_0}(\Omega,|x|^{-\beta})})^{\frac{p}{p-q}} + (|\mu| \|f\|_{L^{p_0}(\Omega,|x|^{-\beta})})^{\frac{p}{p-q}} < C^*.$$

Recall $(u_0, v_0) = (e_1u_\epsilon, e_2u_\epsilon)$. It is easy to see that

$$\int_\Omega |x|^{-bp^*} F(x, u_0, v_0)\, dx > 0.$$

Combining this with Lemma 7, from the definition of $\theta^-_{\lambda,\mu}$, (3.12) and (3.15), we obtain that there exists $t_0 > 0$ such that $(t_0u_0, t_0v_0) \in N^-_{\lambda,\mu}$ and

$$\theta^-_{\lambda,\mu} \leq J_{\lambda,\mu}(t_0u_0, t_0v_0) \leq \sup_{t\geq t_0} J_{\lambda,\mu}(t_0u_0, t_0u_0) < c_\infty,$$

for all $0 < (|\lambda|\|f\|_{L^{p_0}(\Omega,|x|^{-\beta})})^{\frac{p}{p-q}} + (|\mu|\|f\|_{L^{p_0}(\Omega,|x|^{-\beta})})^{\frac{p}{p-q}} < C^*$. □

Theorem 4. *If*

$$0 < (|\lambda|\|f\|_{L^{p_0}(\Omega,|x|^{-\beta})})^{\frac{p}{p-q}} + (|\mu|\|f\|_{L^{p_0}(\Omega,|x|^{-\beta})})^{\frac{p}{p-q}} < C^*_0$$

and (H1)–(H3) *hold, then* $J_{\lambda,\mu}$ *has a minimizer* (u^-_0, v^-_0) *in* $N^-_{\lambda,\mu}$ *and it satisfies*

(i) $J_{\lambda,\mu}(u^-_0, v^-_0) = \theta^-_{\lambda,\mu}$,

(ii) (u^-_0, v^-_0) *is a positive solution of* (1.1),

where $C^*_0 = \min\{C^*, \Lambda_0\}$.

Proof. By the Lemma 8(ii), there exists a minimizing sequence $\{(u_n, v_n)\} \subset N^-_{\lambda,\mu}$ in W for $J_{\lambda,\mu}$ for all

$$0 < \left(|\lambda|\|f\|_{L^{p_0}(\Omega,|x|^{-\beta})}\right)^{\frac{p}{p-q}} + \left(|\mu|\|f\|_{L^{p_0}(\Omega,|x|^{-\beta})}\right)^{\frac{p}{p-q}} < C_0.$$

From Lemmas 12, 13 and 6(ii), for

$$0 < \left(|\lambda|\|f\|_{L^{p_0}(\Omega,|x|^{-\beta})}\right)^{\frac{p}{p-q}} + \left(|\mu|\|f\|_{L^{p_0}(\Omega,|x|^{-\beta})}\right)^{\frac{p}{p-q}} < C^*,$$

$J_{\lambda,\mu}$ satisfies $(PS)_{\theta^-_{\lambda,\mu}}$ condition and $\theta^-_{\lambda,\mu} > 0$. Since $J_{\lambda,\mu}$ is coercive on $N_{\lambda,\mu}$, we get that (u_n, v_n) is bounded in W. Therefore, there exists a subsequence still denote by (u_n, v_n) and $(u^-_0, v^-_0) \in N^-_{\lambda,\mu}$ such that $(u_n, v_n) \to (u^-_0, v^-_0)$ strongly in W and $J_{\lambda,\mu}(u^-_0, v^-_0) = \theta^-_{\lambda,\mu} > 0$ for all

$$0 < (|\lambda|\|f\|_{L^{p_0}(\Omega,|x|^{-\beta})})^{\frac{p}{p-q}} + (|\mu|\|f\|_{L^{p_0}(\Omega,|x|^{-\beta})})^{\frac{p}{p-q}} < C^*_0.$$

Finally, by the same arguments as in the proof of Theorem 3, for all

$$0 < (|\lambda|\|f\|_{L^{p_0}(\Omega,|x|^{-\beta})})^{\frac{p}{p-q}} + (|\mu|\|f\|_{L^{p_0}(\Omega,|x|^{-\beta})})^{\frac{p}{p-q}} < C^*_0,$$

we have that (u^-_0, v^-_0) is a positive solution of problem (1.1). □

Now, we complete the proof of Theorem 1 and Theorem 2: By Theorem 3, we obtain that for all $0 < (|\lambda|\|f\|_{L^{p_0}(\Omega,|x|^{-\beta})})^{\frac{p}{p-q}} + (|\mu|\|f\|_{L^{p_0}(\Omega,|x|^{-\beta})})^{\frac{p}{p-q}} < C$, problem (1.1) has a positive solution $(u^+_0, v^+_0) \in N^+_{\lambda,\mu}$. On the other hand, from Theorem 4, we get the second positive solution $(u^-_0, v^-_0) \in N^-_{\lambda,\mu}$ for all

$$0 < (|\lambda|\|f\|_{L^{p_0}(\Omega,|x|^{-\beta})})^{\frac{p}{p-q}} + (|\mu|\|f\|_{L^{p_0}(\Omega,|x|^{-\beta})})^{\frac{p}{p-q}} < C^*_0 < C.$$

Since $N^+_{\lambda,\mu} \cap N^-_{\lambda,\mu} = \emptyset$, this implies that (u^+_0, v^+_0) and (u^-_0, v^-_0) are distinct. This completes the proofs of Theorem 1 and Theorem 2.

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