Vibration Control of a Viscoelastic Flexible Marine Riser with Vessel Dynamics

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Abstract. In this work, we investigate the asymptotic behavior of solutions of a viscoelastic flexible marine riser with vessel dynamics. Under a suitable control applied at the top end of the riser, we establish explicit decay rates for a large class of relaxation functions. In particular, exponentially and polynomially (or power type) decaying functions are included in this class. Our method is based on the multiplier technique. Numerical simulations justifying the effectiveness of the proposed boundary control to suppress the vibrations of the flexible marine riser are provided.

Keywords: stability, vibration control, flexible marine riser, boundary control, Euler-Bernoulli beam structure, viscoelasticity.

AMS Subject Classification: 35L20; 93D15; 93D20.

1 Introduction

When a cylinder is submerged in a moving fluid we assist to the formation of complex vertices which destabilize the cylinder from its initial position. They

make the cylinder vibrate. This phenomenon is called Vortex-induced vibra-
tions. It is one of the main concern in engineering and in particular in offshore
industry. Companies are spending a lot of money in their investments to rem-
edy to this problem. Indeed, vortices produce unwanted and in fact destructive
vibrations of the pipeline connecting the well in the bottom of the sea to the
floating platform. In addition to these "transversal" vibrations and as the riser
transports crude oil or natural gaz there will be internal pressure due to the
fluid flow (often with pulses). There are also other causes of vibrations like
transmission of waves, earthquake ground motions, etc.

An immediate consequence of these vibrations is the partial or complete
destruction of the riser, or at least a limitation of its lifetime. This damage has
to be predicated long before. A failure will affect the productivity or lead to
an ecological disaster in case of leakage. Therefore there is a need for periodic
inspections and maintenance.

Several ways have been devised to improve the efficiency and the lifespan
of the riser. In particular, different forms, shapes and material have been used
to suppress or at least reduce the unwanted vibrations. Also, some additional
devices have been added to the structure like fairings, streaks and tensioner
to increase the stiffness or the riser. Yet these devices are not able to give
satisfactory results. Researchers have been forced to use sensors and actuators
to control the vibrations. Unless the platform is fixed and the riser is clamped,
for instance, to the supports (legs) of the structure, it is not practical to place
these devices in inaccessible spots. Unfortunately, most of the platforms are
floating because of the deep water (up to 3 km) and the need for mobility. The
best location for control is the top of the riser. That is a boundary control
at only one endpoint. This direction of research has gained a lot of attention.
A successful boundary control should stabilize the system quickly enough to
avoid any damage. We refer the reader to [4, 6, 8, 13, 17, 23] where the authors
studied the boundary control of flexible marine riser. It is worth mentioning
the work of Seghour *et al.* [29] where the authors considered the problem

$$\rho w_{tt}(x,t) + EI w_{xxxx}(x,t) - EI \int_0^t h(t-s) w_{xxxx}(x,s)\, ds - T w_{xx}(x,t) = 0, \quad (1.1)$$

for all $(x,t) \in (0,L) \times [0,\infty)$, with the boundary conditions and the initial data

$$\begin{cases} w(0,t) = w_x(0,t) = w_{xx}(L,t) = 0, \quad t \geq 0, \\ -EI w_{xxx}(L,t) + EI \int_0^t h(t-s) w_{xxx}(L,s)\, ds + T w_x(L,t) \\ \quad = u(t) - d_s w_t(L,t) - M_s w_{tt}(L,t), \ t \geq 0, \\ w(x,0) = w_0(x), \quad w_t(x,0) = w_1(x), \quad x \in [0,L]. \end{cases} \quad (1.2)$$

Here $w(x,t)$, $w_{tt}(x,t)$, $w(L,t)$ and $w_t(L,t)$ are the displacement, accelera-
tion of the riser at the position x for the time t, the position and velocity of the
vessel at L, respectively. The positive constants ρ, L, EI and T represent, the
uniform mass per unit length, the length, the bending stiffness and the tension
of the riser, respectively. The coefficients M_s and d_s denote the mass of the
surface vessel and the vessel damping, and $u(t)$ is the top boundary control of

the riser. The integral term in the Equation (1.1) represents the viscoelastic damping term. This term appears in the constitutive relationship between the stress and the strain according to the Boltzmann Principle (see [7]). The kernel h is called the relaxation function. The expressions $w_0(x)$ and $w_1(x)$ are given initial data. Under the following control

$$u(t) = \frac{-K}{w_t(L,t)}\left[w_{xxx}(L,t)^2 + w(L,t)^2 + \left(\int_0^t h(t-s)w_{xxx}(L,s)\,ds\right)^2\right], \ t \geq 0,$$

where K is positive constant, for $w_t(L,t) \neq 0$. The authors proved an exponential decay result for solutions of the problem (1.1)–(1.2) for kernels h verifying the condition

$$0 < h'(t) + \gamma h(t) \leq \xi(t), \quad t \geq 0$$

for some positive constant γ and nonnegative function $\xi(t)$.

Also, in [12] the authors studied a similar problem to (1.1)–(1.2) when $h \equiv 0$ in (1.1) with internal damping of the form $w_t(x,t)$.

The boundary control has received great attention in recent years due its numerous applications in engineering such as vibration control of flexible robotic manipulator [2,3,16], the flexible wings of a robotic aircraft [14], the control of a nonuniform gantry crane [11], the industrial moving strip [15], the nonlinear axially moving string [19] and the references therein. The boundary stabilization and control has been developed for the Euler-Bernoulli beam, by among others: Krall [20], de Querioz *et al.* [5], Li *et al.* [21], Andrews *et al.* [1], Guo and Huang [10], Guo and Guo [9] (see also the references therein). In particular, dissipative mechanisms of viscoelastic type have been implemented. Park *et al.* [28] looked into the following problem

$$w_{tt}(x,t) + w_{xxxx}(x,t) - \int_0^t h(t-\tau)w_{xxxx}(x,\tau)d\tau + g(w_t(x,t)) = 0,$$

where $(x,t) \in [0,L] \times (0,\infty)$, with the boundary conditions and initial data

$$\begin{cases} w(0,t) = w_x(0,t) = w_{xx}(L,t) = w_{xxx}(0,t) = 0, \quad t \geq 0, \\ w_{xxx}(L,t) - \int_0^t h(t-\tau)w_{xxx}(L,t)d\tau = f(y(L,t)), \ t \geq 0, \\ w(x,0) = w_0(x), \ w_t(x,0) = w_1(x) \ x \in [0,L]. \end{cases}$$

They proved the existence of solutions using the Faedo-Galerkin method and investigated the exponential stability of the system by using the multiplier technique for kernels h of an exponential type, i.e.

$$\exists k_i > 0, \ i = 0, ..., 2 : -k_0 h(t) \leq h_t(t) \leq -k_1 h(t), \quad 0 < h_{tt}(t) \leq k_2 h(t).$$

A similar result was obtained by the same authors in [27] together with a boundary output feedback control

$$\begin{cases} w(0,t) = w_x(0,t) = w_{xx}(L,t) = 0, \quad t \geq 0, \\ w_{xxx}(L,t) - \int_0^t g(t-\tau)w_{xxx}(L,t)d\tau = u(t), \ t \geq 0, \\ w_{out} = w_t(L,t) \quad t \geq 0, \end{cases}$$

where w_{out} is the measured signal of the system at time t with

$$\begin{cases} u(t) = k(t)w_t(L,t), & k(0) > 0, \quad t \geq 0, \\ k'(t) = rw_t^2(L,t), & t \geq 0. \end{cases}$$

An analogous result was also established by Kang *et al.* [18] with the boundary conditions

$$\begin{cases} w(0,t) = w_x(0,t) = w_{xx}(L,t) = 0, & t \geq 0, \\ w_{xxx}(L,t) - \int_0^t g(t-\tau)w_{xxx}(L,t)d\tau = u(t) - \tilde{\theta}\sin(t), & t \geq 0, \\ w_{out} = y_t(L,t) & t \geq 0, \end{cases}$$

where $\tilde{\theta}$ is a positive constant. For exponentially decaying kernels, using the multiplier technique they proved the exponential stability under the following adaptive output feedback controller

$$\begin{cases} u(t) = h(t)w_t(L,t) + \theta(t)\sin t, & t \geq 0, \\ h_t(t) = rw_t^2(L,t), & h(0) = h_0 > 0, \quad t \geq 0, \quad r > 0, \\ \theta_t(t) = w_t(L,t)\sin t, & \theta(0) = \theta_0. \end{cases}$$

The main objective of this work, is to establish different types of decay rates of the system (1.1)–(1.2) under a suitable control acting on top of the riser. This will be shown for a large class of kernels. Namely, we suppose that $h'(t) \leq \xi(t)$ for some non-negative function $\xi(t)$ and the kernel h verifies

$$h(t-s) \geq \mu(t)\int_t^\infty h(\sigma - s)\,d\sigma, \; t > 0$$

for some function $\mu(t)$. This condition is satisfied by a large class of functions including polynomials and exponentials (see Remark 2). It is important to consider these conditions which allow the use of more materials that could be more convenient. For other types of kernels, one can consult [2,19].

The well-posedness of our system (1.1)–(1.2) can be established by using Faedo-Galerkin method, we refer the reader to [3]. We set

$$\mathcal{V} = \{w \in H^2(0,L), \; w(0) = w_x(0) = 0\}, \quad \mathcal{W} = \{w \in H^4(0,L), \; w_{xx}(L) = 0\},$$

where $H^2(0,L)$ and $H^4(0,L)$ are the usual Sobolev spaces.

Proposition 1. *Let $w_0 \in \mathcal{W}$, $w_1 \in \mathcal{V}$ and $h(t)$ be a nonnegative summable kernel. Under the control $u(t)$ defined in (3.1), the system (1.1)–(1.2) has a unique solution w in the class*

$$w \in L^\infty([0,T); \mathcal{W}), \; w_t \in L^\infty([0,T); \mathcal{V}), \; w_{tt} \in L^\infty([0,T); L^2(0,L)),$$

where $T > 0$.

The rest of our paper is organized as follows. In Section 2 we prepare some lemmas which will be useful in the proof of our result. The arbitrary decay result under a suitable boundary control $u(t)$ is stated and proved in Section 3. Simulations are presented to illustrate the performance of the proposed control in Section 4, and some conclusions are given in Section 5.

2 Preliminaries results

In this section, we give some lemmas which will be used throughout this paper. First, to simplify the notation, we denote by $\square$ the binary operator, defined by

$$(h\square v)(t) = \int_0^L \int_0^t h(t-s)\Big(v(t) - v(s)\Big)^2 ds dx, \quad t \geq 0.$$

We shall use the following inequalities.

Lemma 1. *Under our boundary conditions* (1.2), *we have*

$$w^2(x,t) \leq L\|w_x\|_2^2, \quad w^2(x,t) \leq L^3\|w_{xx}\|_2^2, \quad w_x^2(x,t) \leq L\|w_{xx}\|_2^2,$$

for all $(x,t) \in [0,L] \times \mathbb{R}^+$ *and*

$$\|w\|_2^2 \leq L^2\|w_x\|_2^2 \leq L^4\|w_{xx}\|_2^2 \quad \forall x \in [0,L].$$

To preserve the hyperbolicity of our system (1.1)–(1.2), we assume that the kernel h satisfies

$$1 - \int_0^\infty h(s)\,ds =: 1 - \kappa > 0.$$

Lemma 2. *The energy functional* $\mathcal{E}(t)$ *of the problem* (1.1)–(1.2) *given by*

$$2\mathcal{E}(t) = \rho\|w_t\|_2^2 + EI\left(1 - \int_0^t h(s)\,ds\right)\|w_{xx}\|_2^2 + T\|w_x\|_2^2$$
$$+ M_s w_t^2(L,t) + EI(h\square w_{xx}),$$

satisfies, for all $t \geq 0$

$$\frac{d}{dt}\mathcal{E}(t) = -\frac{EI}{2}h(t)\|w_{xx}\|_2^2 + \frac{EI}{2}(h'\square w_{xx}) - d_s w_t^2(L,t) + u(t)w_t(L,t), \quad (2.1)$$

where $\|.\|_2$ *is the norm in* $L^2(0,L)$.

Proof. By multiplying the Equation (1.1) by w_t, integrating over $(0,L)$ and using the boundary conditions (1.2), we get

$$\frac{1}{2}\frac{d}{dt}\Big(\rho\|w_t\|_2^2 + EI\|w_{xx}\|_2^2 + T\|w_x\|_2^2 + M_s w_t^2(L,t)\Big)$$
$$= EI\int_0^L w_{xxt}\int_0^t h(t-s)w_{xx}(s)dsdx - d_s w_t^2(L,t) + u(t)w_t(L,t), \quad t \geq 0.$$

Now, it suffices to observe that

$$2\int_0^L \omega_{xxt}\int_0^t h(t-s)\omega_{xx}(s)dsdx = \int_0^L(h'\square w_{xx})dx - h(t)\|w_{xx}\|_2^2$$
$$- \frac{d}{dt}\Big[\int_0^L(h\square w_{xx})dx - \Big(\int_0^t h(s)ds\Big)\|w_{xx}\|_2^2\Big],$$

for all $t \geq 0$, to conclude. $\square$

From this lemma we see that it is not clear that the energy $\mathcal{E}(t)$ of our system (1.1)–(1.2) is decreasing. Now, we set

$$V(t) = \mathcal{E}(t) + \sum_{i=1}^{4} \lambda_i V_i(t), \quad t \geq 0, \tag{2.2}$$

where λ_i, $i = 1, ..., 4$ are positive constants which will be chosen later, and

$$V_1(t) = \rho \int_0^L w_t w \, dx + M_s w_t(L,t) w(L,t), \quad t \geq 0,$$

$$V_2(t) = -\rho \int_0^L w_t \int_0^t h(t-s) \Big(w(t) - w(s) \Big) ds \, dx$$

$$\qquad - M_s w_t(L,t) \int_0^t h(t-s) \Big(w(L,t) - w(L,s) \Big) ds, \quad t \geq 0,$$

$$V_3(t) = \int_0^t \left(\int_t^\infty h(\sigma - s) \, d\sigma \right) \|w_{xx}(s)\|_2^2 \, ds, \quad t \geq 0$$

$$V_4(t) = \int_0^t \left(\int_t^\infty \xi(\sigma - s) \, d\sigma \right) \|w_{xx}(s)\|_2^2 \, ds, \quad t \geq 0.$$

Remark 1. The functionals V_i, $i = 1, ..., 4$ should be chosen in such a way that their derivatives will provide us with similar terms to the ones in the energy expression (kinetic, potential ...) but with negative coefficients. We will control these terms and have the energy with a negative sign in the right hand side of the estimate of the derivative of (2.2).

The first result tells us that $V(t)$ and $\mathcal{E}(t) + V_3(t) + V_4(t)$ are equivalent.

Lemma 3. *There exist $\rho_i > 0, i = 1, 2$ such that*

$$\rho_1 \Big[\mathcal{E}(t) + V_3(t) + V_4(t) \Big] \leq V(t) \leq \rho_2 \Big[\mathcal{E}(t) + V_3(t) + V_4(t) \Big],$$

for all $t \geq 0$.

Proof.   By Young's and Hölder's inequalities, we obtain

$$V_1(t) \leq \frac{\rho}{2} \|w_t\|_2^2 + \frac{\rho L^4}{2} \|w_{xx}\|_2^2 + \frac{M_s}{2} w^2(L,t) + \frac{M_s}{2} w_t^2(L,t), \quad t \geq 0.$$

Therefore, by using Lemma 1, we get

$$V_1(t) \leq \frac{\rho}{2} \|w_t\|_2^2 + \left(\rho L + M_s \right) \frac{L^3}{2} \|w_{xx}\|_2^2 + \frac{M_s}{2} w_t^2(L,t), \quad t \geq 0. \tag{2.3}$$

For $V_2(t)$, it easy to see

$$V_2(t) \leq \frac{\rho}{2} \|w_t\|_2^2 + \frac{\rho L^4 \kappa}{2} (h \square w_{xx}) + \frac{M_s}{2} w_t^2(L,t)$$

$$\qquad + \frac{M_s \kappa}{2} \int_0^t h(t-s) \big(w(L,t) - w(L,s) \big)^2 ds, \quad t \geq 0.$$

Using Lemma 1 again, we have

$$\int_0^t h\left(t-s\right)\left(w\left(L,s\right)-w\left(L,t\right)\right)^2 ds \le L^3\left(h\Box w_{xx}\right),\ t \ge 0.$$

The last relation, imply that

$$V_2\left(t\right) \le \frac{\rho}{2}\|w_t\|_2^2 + \left(\rho L + M_s\right)\frac{\kappa L^3}{2}\left(h\Box w_{xx}\right) + \frac{M_s}{2}w_t^2\left(L,t\right),\ t \ge 0. \tag{2.4}$$

Gathering (2.3) and (2.4), we end up with

$$
\begin{aligned}
V\left(t\right) \le &\frac{\rho}{2}\left(1+\lambda_1+\lambda_2\right)\|w_t\|_2^2 + \frac{1}{2}\left[EI + \lambda_2\left(\rho L + M_s\right)\kappa L^3\right]\left(h\Box w_{xx}\right)\left(t\right)\\
&+ \frac{1}{2}\left[EI\left(1-\int_0^t h\left(s\right)ds\right) + \lambda_1\left(\rho L + M_s\right)L^3\right]\|w_{xx}\|_2^2\\
&+ \frac{T}{2}\|w_x\|_2^2 + \frac{1}{2}M_s\left(1+\lambda_1+\lambda_2\right)w_t^2\left(L,t\right) + \lambda_3 V_3\left(t\right) + \lambda_4 V_4\left(t\right),\ t \ge 0.
\end{aligned}
$$

Therefore,

$$V\left(t\right) \le \rho_2\Big(\mathcal{E}\left(t\right) + \sum_{j=3}^4 V_j(t)\Big),\ t \ge 0$$

for some positive constant ρ_2. Similarly,

$$
\begin{aligned}
2V\left(t\right) \le &\rho\left(1-\lambda_1-\lambda_2\right)\|w_t\|_2^2 + \left[EI - \lambda_2\left(\rho L + M_s\right)\kappa L^3\right]\left(h\Box w_{xx}\right)\\
&+ \left[EI\left(1-\kappa\right) - \lambda_1\left(\rho L + M_s\right)L^3\right]\|w_{xx}\|_2^2 + T\|w_x\|_2^2\\
&+ M_s\left(1-\lambda_1-\lambda_2\right)w_t^2\left(L,t\right) + 2\lambda_3 V_3\left(t\right) + 2\lambda_4 V_4\left(t\right),\ t \ge 0.
\end{aligned}
$$

Hence,

$$V\left(t\right) \ge \rho_2\Big(\mathcal{E}\left(t\right) + \sum_{j=3}^4 V_j(t)\Big),\quad t \ge 0$$

for some constant $\rho_1 > 0$, provided that

$$\lambda_1 < \min\left[1, \frac{EI\left(1-\kappa\right)}{\left(\rho L + M_s\right)L^3}\right],\ \lambda_2 < \min\left[1-\lambda_1, \frac{EI}{\left(\rho L + M_s\right)\kappa L^3}\right].$$

□

The following inequality will be used repeatedly in the sequel.

Lemma 4. *We have*

$$\delta\gamma \le \eta\delta^2 + \frac{\gamma^2}{4\eta},\quad \delta,\gamma \in \mathbb{R},\quad \eta > 0.$$

The identity to follow is easy to justify and is helpful to prove our results.

Lemma 5. *(See [30]) We have for $h \in C(0, \infty)$ and $y \in C\left((0, \infty); L^2(0, L)\right)$*

$$
\int_0^L y \int_0^t h(t-s) y(s) \, ds dx = \frac{1}{2} \left(\int_0^t h(s) \, ds \right) \|y\|_2^2 - \frac{1}{2} (h \square y)(t)
$$
$$
+ \frac{1}{2} \int_0^t h(t-s) \|y(s)\|_2^2 \, ds, \tag{2.5}
$$

for all $t \geq 0$.

3   Asymptotic behavior

Our goal in this section is to prove a stabilization result. We start by giving some notions and assumptions used in this work in line with [30]. Let $\mathcal{A}$ be a measurable set, such that $\mathcal{A} \subset \mathbb{R}^+$, we denote by $\hat{h}(\mathcal{A})$ the probability measure

$$
\hat{h}(\mathcal{A}) = \frac{1}{\kappa} \int_{\mathcal{A}} h(s) \, ds.
$$

The non-decreasingness set of h is defined by $Q_h := \{s \in \mathbb{R}^+ : h'(s) \geq 0\}$, we also define the non-decreasingness rate of h by $\mathcal{R}_h := \hat{h}(Q_h)$. We suppose that the kernel $h(t)$ verifies (see [30]):

(H1) $h(t) \geq 0$ for all $t \geq 0$ and $0 < \kappa = \int_0^{+\infty} h(s) \, ds < 1$.

(H2) $h(t)$ is absolutely continuous and of bounded variation on $(0, \infty)$ and $h'(t) \leq \xi(t)$ for some non-negative summable function $\xi(t) \, (= \max\{0, h'(t)\})$ and almost all $t > 0$.

(H3)   There exist two functions $\mu_i(t) > 0$, $i = 1, 2$ such that

$$
h(t-s) \geq \mu_1(t) \int_t^\infty h(\sigma - s) \, d\sigma, \ t > 0,
$$
$$
\xi(t-s) \geq \mu_2(t) \int_t^\infty \xi(\sigma - s) \, d\sigma, \ t > 0.
$$

Remark 2. Notice that this assumption (H3) is satisfied by a large class of functions namely functions of polynomial type and functions of exponential type. Indeed, $\xi(t) = (1+t)^{-\alpha}$, $\alpha > 0$ satisfies the assumption with $\mu(t) = (\alpha - 1)/(1+t)$. The function $\xi(t) = e^{-\beta t}$, $\beta > 0$ satisfies the assumption with $\mu(t) = \beta$.

Let $t_* > 0$ be a number such that $\int_0^{t_*} h(s) ds = h_* > 0$. For simplicity, we consider kernels continuous everywhere and continuously differentiable a.e.

In order to stabilize the system (1.1)–(1.2), we propose the following boundary control

$$
u(t) = d_s w_t(L, t), \quad t \geq 0. \tag{3.1}
$$

Remark 3. The boundary control law defined in (3.1), is proposed to suppress the vibrations of the flexible marine riser in a fast manner. There are other ways how to design suitable boundary controllers for PDEs in the literature, the reader is referred to [22, 24, 25, 26], and their references.

Now, we are ready to state and prove our main result.

Theorem 1. *Under the assumptions* (H1)–(H3) *and the control $u(t)$ defined in* (3.1), *if $\mathcal{R}_h$ is sufficiently small, then, for small L^1-norm of ξ there exist positive constants Λ and ν such that*

(a) *if* $\lim\limits_{t\longrightarrow\infty} \mu_1(t) = 0$ *or* $\lim\limits_{t\longrightarrow\infty} \mu_2(t) = 0$,

$$\mathcal{E}(t) \leq \Lambda \exp\left(-\nu \int_0^t \min\left\{\mu_1(s),\ \mu_2(s)\right\}ds\right),\ t \geq 0.$$

(b) *If* $\lim\limits_{t\longrightarrow\infty} \mu_1(t) \neq 0$ *and* $\lim\limits_{t\longrightarrow\infty} \mu_2(t) \neq 0$, *we have*

$$\mathcal{E}(t) \leq \Lambda e^{-\nu t},\ t \geq 0.$$

Proof. A differentiation of $V_1(t)$ with respect to t along solutions of (1.1) gives

$$\frac{d}{dt}V_1(t) = \rho\|w_t\|_2^2 + M_s w_t(L,t)^2 + M_s w_{tt}(L,t)w(L,t) \\ + I_1 + I_2 + I_3, \quad t \geq 0, \tag{3.2}$$

where

$$I_1 = -EI \int_0^L ww_{xxxx}dx, \quad I_2 = EI \int_0^L w \int_0^t h(t-s)w_{xxxx}(s)\,dsdx,$$

$$I_3 = T \int_0^L ww_{xx}dx.$$

Integrating by parts in I_i $i = 1, ..., 3$ and using the boundary conditions (1.2), we get

$$I_1 = -EIw(L,t)w_{xxx}(L,t) - EI\|w_{xx}\|_2^2, \quad t \geq 0, \tag{3.3}$$

$$I_2 = EIw(L,t)\int_0^t h(t-s)w_{xxx}(L,s)\,ds \\ + EI \int_0^L w_{xx}\int_0^t h(t-s)w_{xx}(s)\,dsdx, \quad t \geq 0, \tag{3.4}$$

$$I_3 = Tw_x(L,t)w(L,t) - T\|w_x\|_2^2, \quad t \geq 0. \tag{3.5}$$

Substituting the estimates (3.3)–(3.5) in Equation (3.2), taking into account the boundary conditions (1.2), the expression of the control $u(t)$ and canceling common terms, we obtain

$$\frac{d}{dt}V_1(t) = \rho\|w_t\|_2^2 + M_s w_t^2(L,t) - EI\|w_{xx}\|_2^2 - T\|w_x\|_2^2 \\ + EI \int_0^L w_{xx}\int_0^t h(t-s)w_{xx}(s)\,dsdx, \quad t \geq 0$$

and by the identity (2.5) in Lemma 5 we get

$$
\begin{aligned}
\frac{d}{dt}V_1(t) &\leq \rho\|w_t\|_2^2 + M_s w_t^2(L,t) - EI\left(1-\frac{\kappa}{2}\right)\|w_{xx}\|_2^2 - T\|w_x\|_2^2 \\
&\quad - \frac{EI}{2}(h\square w_{xx}) + \frac{EI}{2}\int_0^t h(t-s)\|w_{xx}(s)\|_2^2 ds, \quad t \geq 0.
\end{aligned}
\tag{3.6}
$$

For the functional $V_2(t)$, we have

$$
\begin{aligned}
\frac{d}{dt}V_2(t) &= -\rho\left(\int_0^t h(s)\,ds\right)\|w_t\|_2^2 - M_s\left(\int_0^t h(s)\,ds\right)w_t^2(L,t) \\
&\quad - \rho\int_0^L w_t \int_0^t h'(t-s)\Big(w(t)-w(s)\Big)dsdx - M_s w_t(L,t) \\
&\quad \times \int_0^t h'(t-s)\Big(w(L,t)-w(L,s)\Big)ds - M_s w_{tt}L,t) \\
&\quad \times \int_0^t h(t-s)\Big(w(L,t)-w(L,s)\Big)ds + I_4 + I_5 + I_6 + I_7, \ t \geq 0,
\end{aligned}
\tag{3.7}
$$

where

$$
I_4 = EI\int_0^L w_{xxxx}\int_0^t h(t-s)\Big(w(t)-w(s)\Big)dsdx,
$$

$$
I_5 = -EI\int_0^L\left(\int_0^t h(t-s)w_{xxxx}(s)\,ds\right)\int_0^t h(t-s)\big(w(t)-w(s)\big)dsdx,
$$

$$
I_6 = -T\int_0^L w_{xx}\int_0^t h(t-s)\big(w(t)-w(s)\big)dsdx.
$$

An integration by parts of I_i, $i = 4,5,6$ and using our boundary conditions (1.2), we obtain for all $t \geq 0$,

$$
\begin{aligned}
I_4 &= EI w_{xxx}(L,t)\int_0^t h(t-s)\big(w(L,t)-w(L,s)\big)ds \\
&\quad + EI\int_0^L w_{xx}\int_0^t h(t-s)\big(w_{xx}(t)-w_{xx}(s)\big)dsdx,
\end{aligned}
\tag{3.8}
$$

$$
\begin{aligned}
I_5 &= -EI\left(\int_0^t h(t-s)w_{xxx}(L,s)ds\right)\int_0^t h(t-s)\big(w(L,t)-w(L,s)\big)ds \\
&\quad - EI\int_0^L\left(\int_0^t h(t-s)w_{xx}(s)ds\right)\int_0^t h(t-s)\big(w_{xx}(t)-w_{xx}(s)\big)dsdx
\end{aligned}
\tag{3.9}
$$

and

$$
\begin{aligned}
I_6 &= -Tw_x(L,t)\int_0^t h(t-s)\big(w(L,t)-w(L,s)\big)ds \\
&\quad + T\int_0^L w_x\int_0^t h(t-s)\big(w_x(t)-w_x(s)\big)dsdx.
\end{aligned}
\tag{3.10}
$$

Collecting the estimates (3.8)–(3.10) in (3.7), using again the boundary conditions, the expression of the control $u(t)$ after canceling common terms, we see that

$$
\frac{d}{dt}V_2(t) = -\left(\int_0^t h(s)\,ds\right)\left(\rho\,\|w_t\|_2^2 + M_s w_t^2(L,t)\right)
$$

$$
-\rho\int_0^L w_t \int_0^t h'(t-s)\left(w(t)-w(s)\right)dsdx - M_s w_t(L,t)\int_0^t h'(t-s)
$$

$$
\times\left(w(L,t)-w(L,s)\right)ds + EI\int_0^L\left|\int_0^t h(t-s)\left(w_{xx}(t)-w_{xx}(s)\right)ds\right|^2 dx
$$

$$
+T\int_0^L w_x\int_0^t h(t-s)\left(w_x(t)-w_x(s)\right)dsdx + EI\left(1-\int_0^t h(s)\,ds\right)
$$

$$
\times\int_0^L w_{xx}\int_0^t h(t-s)\left(w_{xx}(t)-w_{xx}(s)\right)dsdx, \quad t\geq 0. \tag{3.11}
$$

Now, we estimate the terms in (3.11). For all measurable sets $\mathcal{A}$ and Q such that $\mathcal{A}=\mathbb{R}^+\backslash Q$, we start by the last term

$$
\int_0^L w_{xx}\int_0^t h(t-s)\left(w_{xx}(t)-w_{xx}(s)\right)dsdx
$$

$$
=\int_0^L w_{xx}\int_{\mathcal{A}\cap[0,t]} h(t-s)\left(w_{xx}(t)-w_{xx}(s)\right)dsdx
$$

$$
+\int_0^L w_{xx}\int_{Q\cap[0,t]} h(t-s)\left(w_{xx}(t)-w_{xx}(s)\right)dsdx
$$

$$
\leq\int_0^L w_{xx}\int_{\mathcal{A}\cap[0,t]} h(t-s)\left(w_{xx}(t)-w_{xx}(s)\right)dsdx
$$

$$
+\left(\int_{Q\cap[0,t]} h(t-s)\,ds\right)\|w_{xx}\|_2^2 - \int_0^L w_{xx}\int_{Q\cap[0,t]} h(t-s)\,w_{xx}(s)\,dsdx, \tag{3.12}
$$

for all $t\geq 0$. Clearly, for $\eta_1 > 0$, we have

$$
\int_0^L w_{xx}\int_{\mathcal{A}_t} h(t-s)\left(w_{xx}(t)-w_{xx}(s)\right)dsdx
$$

$$
\leq \eta_1\|w_{xx}\|_2^2 + \frac{\kappa}{4\eta_1}\int_0^L\int_{\mathcal{A}_t} h(t-s)\left(w_{xx}(t)-w_{xx}(s)\right)^2 dsdx,
$$

where we have used the notation $\mathcal{B}_t = \mathcal{B}\cap[0,t]$ and

$$
\int_0^L w_{xx}\int_{Q_t} h(t-s)\,w_{xx}(s)\,dsdx
$$

$$
\leq\frac{1}{2}\left(\int_{Q_t} h(t-s)\,ds\right)\|w_{xx}\|_2^2 + \frac{1}{2}\int_{Q_t} h(t-s)\,\|w_{xx}(s)\|_2^2\,ds, \quad t\geq 0.
$$

These two estimations show that (3.12), becomes

$$\int_0^L w_{xx} \int_0^t h(t-s)\left(w_{xx}(t)-w_{xx}(s)\right)dsdx \leq \left(\eta_1 + \frac{3}{2}\int_{Q_t} h(t-s)\,ds\right)$$

$$\times \|w_{xx}\|_2^2 + \frac{\kappa}{4\eta_1}\int_0^L \int_{\mathcal{A}_t} h(t-s)\left(w_{xx}(t)-w_{xx}(s)\right)^2 dsdx$$

$$+ \frac{1}{2}\int_{Q_t} h(t-s)\|w_{xx}(s)\|_2^2\,ds, \quad \eta_1 > 0, \quad t \geq 0. \tag{3.13}$$

For the 5th term, it holds that for $\eta_2 > 0$

$$\int_0^L \left|\int_0^t h(t-s)\left(w_{xx}(t)-w_{xx}(s)\right)ds\right|^2 dx \leq \left(1 + \frac{1}{\eta_2}\right)\kappa$$

$$\times \int_0^L \int_{\mathcal{A}_t} h(t-s)\left(w_{xx}(t)-w_{xx}(s)\right)^2 dsdx + (1+\eta_2)\left(\int_{Q_t} h(t-s)\,ds\right)$$

$$\times \int_0^L \int_{Q_t} h(t-s)\left(w_{xx}(t)-w_{xx}(s)\right)^2 dsdx, \quad t \geq 0. \tag{3.14}$$

For the 6th term, using Lemma 4, it is easy to see that

$$\int_0^L w_x \int_0^t h(t-s)\left(w_x(t)-w_x(s)\right)dsdx \leq \eta_3\|w_x\|_2^2 + \frac{\kappa L}{2\eta_3}$$

$$\times \int_0^L \int_{\mathcal{A}_t} h(t-s)\left(w_{xx}(t)-w_{xx}(s)\right)^2 dsdx + \frac{\kappa L}{2\eta_3}\left(\int_{Q_t} h(t-s)\,ds\right)$$

$$\times \int_0^L \int_{Q_t} h(t-s)\left(w_{xx}(t)-w_{xx}(s)\right)^2 dsdx, \quad \eta_3 > t \geq 0. \tag{3.15}$$

Using Lemma 4 again and the assumption (H2), the third and the forth terms in the right-hand side of (3.12) can be estimated for $\eta_4 > 0$ as follows

$$\int_0^L w_t \int_0^t h'(t-s)\left(w(t)-w(s)\right)dxds$$

$$\leq \eta_4\|w_t\|_2^2 + \frac{L^4}{4\eta_4}\left(\int_0^t |h'(s)|\,ds\right)(|h'|\Box w_{xx}) \tag{3.16}$$

$$\leq \eta_4\|w_t\|_2^2 - \frac{L^4}{4\eta_4}BV[h]\int_0^L \int_{\mathcal{A}_t} h'(t-s)\left(w_{xx}(t)-w_{xx}(s)\right)^2 dsdx$$

$$+ \frac{L^4}{4\eta_4}\left(\int_{Q_t} \xi(t-s)\,ds\right)\int_0^L \int_{Q_t} \xi(t-s)\left(w_{xx}(t)-w_{xx}(s)\right)^2 dsdx, \quad t \geq 0$$

and

$$w_t(L,t)\int_0^t h'(t-s)\left(w(L,t)-w(L,s)\right)ds \leq \eta_4 w_t^2(L,t) + \frac{1}{4\eta_4}\left(\int_0^t |h'(s)|\,ds\right)$$

$$\times \int_0^t |h'(t-s)|\left(w(L,t)-w(L,s)\right)^2 ds \leq \eta_4 w_t^2(L,t) - \frac{L^3}{4\eta_4}BV[h]$$

$$\times \int_0^L \int_{\mathcal{A}_t} h'(t-s)\left(w_{xx}(t) - w_{xx}(s)\right)^2 dsdx + \frac{L^3}{4\eta_4}\left(\int_{Q_t} \xi(t-s)\,ds\right)$$

$$\times \int_0^L \int_{Q_t} \xi(t-s)\left(w_{xx}(t) - w_{xx}(s)\right)^2 dsdx, \quad t \geq 0, \tag{3.17}$$

where $BV[h]$ is the total variation of h.

The insertion of (3.13)–(3.17) into Equation (3.11) gives for all $t \geq t_* > 0$

$$\frac{d}{dt}V_2(t) \leq (\eta_4 - h_*)\left(\rho\|w_t\|_2^2 + M_s w_t^2(L,t)\right) + T\eta_3\|w_x\|_2^2 + \frac{EI}{2}(1-h_*)$$

$$\times \int_{Q_t} h(t-s)\|w_{xx}(s)\|_2^2\,ds + EI(1-h_*)\left(\eta_1 + \frac{3}{2}\int_{Q_t} h(t-s)\,ds\right)\|w_{xx}\|_2^2$$

$$+ \left[EI\left(1 + \frac{1-h_*}{4\eta_1} + \frac{1}{\eta_2}\right) + \frac{TL}{2\eta_3}\right]\kappa\int_0^L\int_{\mathcal{A}_t} h(t-s)\left(w_{xx}(s) - w_{xx}(t)\right)^2 dsdx$$

$$+ \left[EI(1+\eta_2) + \frac{L}{2\eta_3}\right]\left(\int_{Q_t} h(t-s)\,ds\right)$$

$$\times \int_0^L\int_{Q_t} h(t-s)\left(w_{xx}(s) - w_{xx}(t)\right)^2 dsdx + \frac{L^3(\rho L + M_s)}{4\eta_4}$$

$$\times \left(\int_{Q_t} \xi(t-s)\,ds\right)\int_0^L\int_{Q_t} \xi(t-s)\left(w_{xx}(s) - w_{xx}(t)\right)^2 dsdx$$

$$- \frac{L^3(\rho L + M_s)}{4\eta_4}BV[h]\int_0^L\int_{\mathcal{A}_t} h'(t-s)\left(w_{xx}(s) - w_{xx}(t)\right)^2 dsdx. \tag{3.18}$$

Next, by using assumption (H3) the differentiation of $V_3(t)$ and $V_4(t)$ (see [30]), yields

$$\frac{d}{dt}V_3(t) \leq \kappa\|w_{xx}\|_2^2 - \eta_5\int_0^t h(t-s)\|w_{xx}(s)\|_2^2\,ds - (1-\eta_5)\mu_1(t)V_3(t) \tag{3.19}$$

and

$$\frac{d}{dt}V_4(t) \leq \left(\int_0^\infty \xi(\sigma)\,d\sigma\right)\|w_{xx}\|_2^2 - \eta_6\int_0^t \xi(t-s)\|w_{xx}(s)\|_2^2\,ds$$
$$- (1-\eta_6)\mu_2(t)V_4(t), \quad t \geq 0, \tag{3.20}$$

for $0 < \eta_5, \eta_6 < 1$. Taking into account the previous estimates (3.6), (3.18), (3.19) and (3.20), and substituting (3.1) in (2.1), we obtain for $t \geq t_* > 0$

$$\frac{d}{dt}V(t) \leq \left[\frac{EI}{2} - \lambda_2\frac{L^3(\rho L + M_s)}{4\eta_4}BV[h]\right]$$

$$\times \int_0^L\int_{\mathcal{A}_t} h'(t-s)\left(w_{xx}(s) - w_{xx}(t)\right)^2 dsdx - \lambda_1\frac{EI}{2}(h\square w_{xx})$$

$$+ \left[\frac{EI}{2} + \lambda_2 \frac{L^3(\rho L + M_s)}{4\eta_4}\left(\int_{Q_t} \xi(t-s)\,ds\right)\right] \int_0^L \int_{Q_t} \xi(t-s)$$

$$\times \left(w_{xx}(s) - w_{xx}(t)\right)^2 ds\,dx + \left[\lambda_1 + \lambda_2(\eta_4 - h_*)\right]\left(\rho\|w_t\|_2^2 + M_s w_t^2(L,t)\right)$$

$$+ \left[\lambda_2 EI(1-h_*)\left(\eta_1 + \frac{3}{2}\int_{Q_t} h(t-s)\,ds\right) + \lambda_3 \kappa + (\lambda_2 \eta_3 - \lambda_1)T\|w_x\|_2^2\right.$$

$$+ \lambda_4 \int_0^\infty \xi(\sigma)\,d\sigma - \lambda_1 EI\left(1 - \frac{\kappa}{2}\right)\Bigg] \|w_{xx}\|_2^2 + \lambda_2\left[EI\left(1 + \frac{1-h_*}{4\eta_1} + \frac{1}{\eta_2}\right) + \frac{TL}{2\eta_3}\right]$$

$$\times \kappa \int_0^L \int_{\mathcal{A}_t} h(t-s)\left(w_{xx}(s) - w_{xx}(t)\right)^2 ds\,dx$$

$$- \lambda_4 \eta_6 \int_0^t \xi(t-s)\|w_{xx}(s)\|_2^2\,ds + \lambda_2\left[EI(1+\eta_2) + \frac{L}{2\eta_3}\right]\left(\int_{Q_t} h(t-s)\,ds\right)$$

$$\times \int_0^L \int_{Q_t} h(t-s)\left(w_{xx}(s) - w_{xx}(t)\right)^2 ds\,dx + \left(\frac{\lambda_1 EI}{2} - \lambda_3 \eta_5\right)$$

$$\times \int_0^t h(t-s)\|w_{xx}(s)\|_2^2\,ds - \lambda_3(1-\eta_5)\mu_1(t)V_3(t)$$

$$+ \frac{\lambda_2 EI}{2}(1-h_*)\int_{Q_t} h(t-s)\|w_{xx}(s)\|_2^2\,ds - \lambda_4(1-\eta_6)\mu_2(t)V_4(t). \quad (3.21)$$

As in [30], we introduce the sets

$$\mathcal{A}_n := \left\{s \in \mathbb{R}^+, nh'(s) + h(s) \le 0\right\}, n \in \mathbb{N},$$

$$\widetilde{\mathcal{A}}_{nt} = \left\{s \in \mathbb{R}^+, 0 \le s \le t,\ nh'(t-s) + h(t-s) \le 0\right\},\ n \in \mathbb{N},$$

$$\widetilde{Q}_{ht} = \left\{s \in \mathbb{R}^+, 0 \le s \le t,\ 0 \le h'(t-s) \le \xi(t-s)\right\}.$$

We notice that $\bigcup_n \mathcal{A}_n = \mathbb{R}^+\backslash\{Q_h \cup \mathcal{N}_h\}$, where

$$Q_h = \left\{s \in \mathbb{R}^+,\ 0 \le h'(s) \le \xi(s)\right\}$$

and $\mathcal{N}_h$ is the null set where h' is not defined. Furthermore, if we denote $Q_n := \mathbb{R}^+\backslash\mathcal{A}_n$, then $\lim_{n\to\to\infty}\hat{h}(Q_n) = \hat{h}(Q_h)$ because $Q_{n+1} \subset Q_n$ for all n and $\bigcap_n \mathcal{F}_n = Q_h \cup \mathcal{N}_h$. We take $\mathcal{A}_t = \widetilde{\mathcal{A}}_{nt}$ and $Q_t = \widetilde{Q}_{nt}$ in (3.21), we choose $\lambda_1 = \lambda_2(h_* - \varepsilon)$ for small $\varepsilon < h_*$, then select λ_2

$$\lambda_2 \le EI\eta_4/\left(L^3(\rho L + M_s)BV[h]\right),$$

so that

$$\frac{EI}{2} - \lambda_2 \frac{L^3(\rho L + M_s)}{4\eta_4}BV[h] \ge \frac{EI}{4},$$

we obtain for all $t \ge t_* > 0$

$$\frac{d}{dt}V(t) \le \lambda_2(\eta_4 - \varepsilon)\left(\rho\|w_t\|_2^2 + M_s w_t^2(L,t)\right) + \left[\lambda_3 \kappa + \lambda_4 \int_0^\infty \xi(\sigma)\,d\sigma\right.$$

$$+ B_1 - \lambda_2(h_* - \varepsilon)EI\left(1 - \frac{\kappa}{2}\right)\Bigg]\|w_{xx}\|_2^2 + \left(\frac{EI(h_* - \varepsilon)}{2}\lambda_2 - \lambda_3 \eta_5\right)$$

$$\times \int_0^t h(t-s)\,\|w_{xx}(s)\|_2^2\,ds + \left\{ \lambda_2 \left[EI\left(1 + \frac{1-h_*}{4\eta_1} + \frac{1}{\eta_2}\right) + \frac{TL}{2\eta_3}\right]\kappa - \frac{EI}{4n}\right\}$$

$$\times \int_0^L \int_{\widetilde{A}_{nt}} h(t-s)\left(w_{xx}(s) - w_{xx}(t)\right)^2 ds dx + \lambda_2\left[\eta_3 - (h_* - \varepsilon)\right]T\|w_x\|_2^2$$

$$+ \lambda_2 \left\{ \left[(1+\eta_2) + \frac{L}{2EI\eta_3}\right]\left(\int_{\widetilde{Q}_{nt}} h(t-s)\,ds\right) - \frac{h_* - \varepsilon}{2}\right\} EI(h\square w_{xx})$$

$$+ \left\{ \frac{3}{4}\left[EI + \lambda_2 \frac{L^3(\rho L + M_s)}{2\eta_4}\int_{\widetilde{Q}_{nt}} \xi(t-\sigma)d\sigma\right] - \lambda_4\eta_6\right\}\int_0^t \xi(t-s)$$

$$\times \|w_{xx}(s)\|_2^2\,ds - \lambda_3(1-\eta_5)\mu_1(t)V_3(t) - \lambda_4(1-\eta_6)\mu_2(t)V_4(t), \quad (3.22)$$

where we have used

$$\int_0^L \int_{\widetilde{Q}_{nt}} \xi(t-s)\left(w_{xx}(s) - w_{xx}(t)\right)^2 ds dx$$

$$\leq \frac{3}{2}\left(\int_{\widetilde{Q}_{nt}} \xi(t-s)\,ds\right)\|w_{xx}\|_2^2 + \frac{3}{2}\int_0^t \xi(t-s)\,\|w_{xx}(s)\|_2^2\,ds,$$

$$B_1 = \frac{3}{4}\left[EI + \lambda_2 \frac{L^3(\rho L + M_s)}{2\eta_4}\int_{\widetilde{Q}_{nt}} \xi(t-s)\,ds\right]\left(\int_{\widetilde{Q}_{nt}} \xi(t-s)\,ds\right)$$

$$+ \lambda_2 EI(1 - h_*)\left(\eta_1 + \frac{3}{2}\int_{\widetilde{Q}_{nt}} h(t-s)\,ds\right).$$

Now, we start selecting the different parameters so that all the coefficients in the right-hand side of (3.22) be negative. First, we select η_1 and η_2 small enough and we choose $\eta_3 = (g_* - \varepsilon)/2$ and $\eta_4 = \varepsilon/2$. If $\mathcal{R}_h$ is sufficiently small, we have

$$\left[(1 + \eta_2) + \frac{L}{2EI\eta_3}\right]\left(\int_{\widetilde{Q}_{nt}} h(t-s)\,ds\right) - \frac{h_* - \varepsilon}{2} < 0,$$

$$\frac{3(1-h_*)}{2}\left(\int_{\widetilde{Q}_{nt}} h(t-s)\,ds\right) < \sigma h_*\left(1 - \frac{\kappa}{2}\right),$$

with

$$\sigma = \frac{3(1-h_*)\kappa}{4(2-\kappa)h_*} + \beta$$

for a small $\beta > 0$, large n and $h_* > 3\kappa/(8-\kappa)$. Moreover we need to select λ_3, so that

$$\frac{1-\varepsilon}{2}EI\lambda_2 \leq \lambda_3\eta_4 \leq \frac{(1-\delta)(h_* - \varepsilon)}{2\kappa}(2-\kappa)EI\lambda_2\eta_4.$$

Once η_1, η_2, n and t_* are fixed, we select also λ_2 such that

$$\lambda_2\left[\left(1 + \frac{1-h_*}{4\eta_1} + \frac{1}{\eta_2}\right) + \frac{TL}{2\eta_3 EI}\right]\kappa - \frac{1}{4n} < 0.$$

Furthermore, we select λ_4 large enough so that

$$\lambda_4 > \frac{3}{4\eta_6}\left[EI + \lambda_2 \frac{L^3(\rho L + M_s)}{2\eta_4}\int_0^\infty \xi(\sigma)d\sigma\right].$$

We need to take $\int_0^\infty \xi(s)\, ds$ small enough to get negative coefficients in the right-hand side of (3.22). Therefore,

$$\frac{d}{dt}V(t) \leq -C_1 \mathcal{E}(t) - C_2 \mu_1(t) V_3(t) - C_3 \mu_2(t) V_4(t)$$

for some positive constants $C_i, i = 1, 2, 3$.

If $\lim_{t\to\infty} \mu_1(t) = 0$ or $\lim_{t\to\infty} \mu_2(t) = 0$ then there exist $\bar{t} \geq t_*$ such that for all $t \geq \bar{t}$ we have

$$\min\Big\{C_2 \mu_1(t), C_3 \mu_2(t)\Big\} \leq C_1.$$

Therefore, by the Lemma 3, we obtain

$$\frac{d}{dt}V(t) \leq -\min\Big\{C_2 \mu_1(t), C_3 \mu_2(t)\Big\} \frac{V(t)}{\rho_2}, \quad t \geq \bar{t}.$$

Then

$$V(t) \leq \delta \exp\left(-\frac{1}{\rho_2} \int_{\bar{t}}^t \min\{C_2 \mu_1(s), C_3 \mu_2(s)\} ds\right), \quad t \geq \bar{t}, \ \ \delta \geq 0.$$

By continuity and Lemma 3 again, it is easy to see

$$\mathcal{E}(t) \leq \Lambda \, exp\left(-\nu \int_{\bar{t}}^t \min\{\mu_1(s), \ \mu_2(s)\} ds\right), \quad t \geq 0$$

for some positive constants Λ and ν.

If $\lim_{t\to\infty} \mu_1(t) \neq 0$ and $\lim_{t\to\infty} \mu_2(t) \neq 0$, then there exist a $\hat{t} \geq t_*$ and $C_4 > 0$ such that $\mu_1(t), \mu_2(t) \geq C_4$ for $t \geq \hat{t}$. Therefore

$$\frac{d}{dt}V(t) \leq -C_5\Big(\mathcal{E}(t) + V_3(t) + V_4(t)\Big) \leq -\frac{C_5}{\rho_2} V(t), \quad C_5 > 0,$$

which leads to

$$\mathcal{E}(t) \leq \Lambda e^{-\nu t}, \ t \geq \hat{t}$$

for some positive constants Λ and ν. By continuity, we obtain a similar estimation on the interval $[0, \max\{\bar{t}, \hat{t}\}]$. $\quad \square$

4   Numerical simulations

In this section, simulations for the presented system described by (1.1) and (1.2) are carried out to illustrate the effectiveness of the proposed control defined in (3.1). The corresponding system parameters are given in the Table 1 and, the initial displacement and velocity are $w_0(x) = \sin(\frac{\pi}{L}x)$ and $w_1(x) = 0$, respectively. The kernel function is chosen to be $h(t) = e^{-kt}$, for some $k \in \mathbb{R}^+$.

We have used finite difference method for numerical approximation of the problem (1.1)–(1.2). Central difference approximations for both spatial and temporal derivatives, in the differential equation (1.1) and the boundary condition (1.2) have been adopted. For the convolution term, we have used the

Table 1. The corresponding system parameters.

Parameter	Description	Value
L	Length	$1000.00m$
ρ	The uniform mass per unit length	$500kg/m$
EI	Bending stiffness	$1.5 \times 10^7 Nm^2$
T	Tension	$8.11 \times 10^7 N$
M_s	Mass of the surface vessel	$9.60 \times 10^6 kg$
d_s	Vessel damping	$1 \times 10^3 Ns/m$

Trapezoid method. We note that, the convolution term requires to store the solution from $t = 0$ and use them at each time step. This, together with large values of the model parameters, demand very small values of time steps, that increases the computational time. Therefore, one can scale the parameters for computer simulations to overcome the time performance. The behavior of the solution of the model problem (1.1)–(1.2), with the control (3.1), have been illustrated in Figure 1 at $x = 500, 800$ and $x = 1000\,m$, respectively.
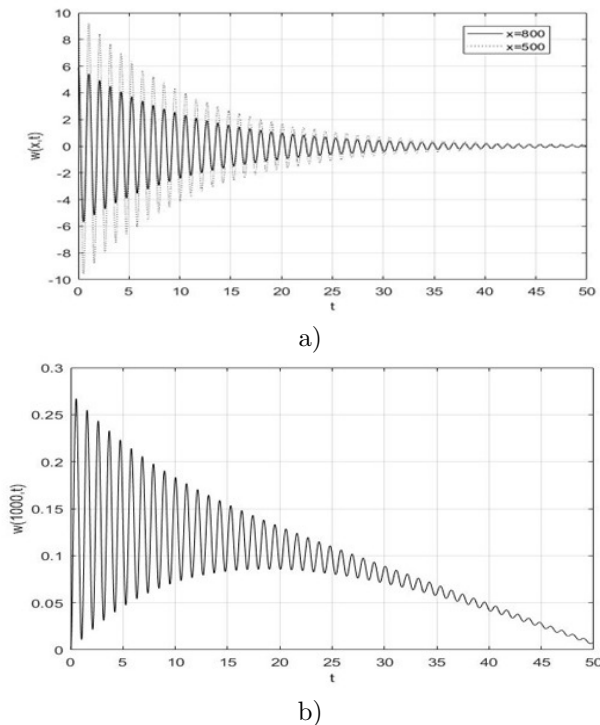

a)

b)

Figure 1. Riser displacement at a) $x = 500$ and $x = 800$, b) $x = 1000$.

In Figure 2, the discrepancy between the solutions with and without the control affect has been shown at three points $x = 500, 800$ and the boundary $x = L$. The affect of the control (3.1) can be seen, after a while due to the
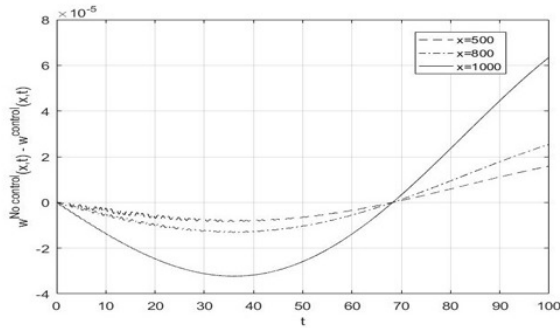
Figure 2. Difference between the riser displacement with/without control, at $x = 500, 800, 1000m$.

convolution term, in particular, when getting closer to the boundary point $x = L$.

5   Conclusions

In this paper, the vibration control of a viscoelastic flexible marine riser with vessel dynamics is studied. A boundary control has been designed to attenuate the vibrations of our system. With the proposed control, closed-looped stability for a large class of relaxation functions has been proven using the multiplier technique. The efficiency of the proposed boundary control is confirmed by some numerical simulations. In a future work, we intend to study the effect of different disturbances, such as the environmental disturbance on the vessel due to the ocean wave, wind and current.

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