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Asymptotic Stability for a Viscoelastic Equation with the Time-Varying Delay

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Abstract. The goal of the present paper is to study the viscoelastic wave equation with the time-varying delay under initial-boundary value conditions. By using the multiplier method together with some properties of the convex functions, the explicit and general stability results of the total energy are proved under the general assumption on the relaxation function g.

Keywords: viscoelasticity, delay term, source term, energy decay.

AMS Subject Classification: 35B40; 26A51; 93D20.

1 Introduction

This paper investigates the following initial-boundary value problem with the time-varying delay

$$\begin{cases} |u_t|^\rho u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t g(t-s)\Delta u(s)ds & \\ \quad +\mu_1 u_t(x,t) + \mu_2 u_t(x,t-\tau(t)) = b|u|^{p-2}u, & (x,t)\in \Omega\times(0,\infty), \\ u_t(x,t-\tau(0)) = f_0(x,t-\tau(0)), & (x,t)\in\Omega\times(0,\tau(0)), \\ u(x,0)=u_0(x),\ u_t(x,0)=u_1(x), & x\in\Omega, \\ u(x,t)=0, & (x,t)\in\partial\Omega\times[0,\infty), \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N (N \geq 1)$ is a bounded domain with a smooth boundary $\partial\Omega$, the unknown $u := u(x,t)$ is a real valued function defined on $\Omega \times (0,\infty)$, ρ, b, μ_1 are positive constants, μ_2 is a real number, $\tau(t)$ represents the positive and non-constant time delay, g is the kernel of the memory term, and the initial data (u_0, u_1, f_0) are given functions belonging to suitable spaces. In addition, the following assumptions are imposed throughout this paper:

($\mathbf{H_1}$) The relaxation function $g : [0,\infty) \to (0,\infty)$ is a differentiable function satisfying

$$1 - \int_0^\infty g(s)ds = l > 0, \tag{1.2}$$

and there exists a C^1 function $G : (0,\infty) \to (0,\infty)$ which is either linear or strictly increasing and strictly convex C^2 function on $(0,r]$, $r \leq g(0)$, with $G(0) = G'(0) = 0$ such that

$$g'(t) \leq -\zeta(t)G(g(t)) \quad \text{for } t \geq 0. \tag{1.3}$$

Here $\zeta(t)$ is a positive non-increasing differentiable function.

($\mathbf{H_2}$) ρ and p satisfy

$$\begin{aligned} &0 < \rho \leq \frac{2}{N-2} \text{ for } N \geq 3 \text{ and } \rho > 0 \text{ for } N = 1,2,\\ &2 < p \leq \frac{2(N-1)}{N-2} \text{ for } N \geq 3 \text{ and } p > 2 \text{ for } N = 1,2. \end{aligned}$$

($\mathbf{H_3}$) The function $\tau \in W^{2,\infty}([0,T])$ for any $T > 0$, and there exists positive constants τ_0 and τ_1 such that

$$0 < \tau_0 \leq \tau \leq \tau_1 \text{ for } t > 0 \text{ and } \tau'(t) \leq d < 1 \text{ for } t > 0.$$

($\mathbf{H_4}$) μ_1 and μ_2 satisfy $|\mu_2| < \frac{2(1-d)}{2-d}\mu_1$.

It is well known that time delay effects which often appear in many practical applications may induce some instabilities. Some results on the local existence and blow-up of solutions to a class of equations with delay have been obtained, the interested readers can refer to [5, 10, 11, 12, 22] and the reference therein. Nicaise and Pignotti [19] considered the wave equation with a delay term in the boundary condition as well as the wave equation with a delayed velocity term and mixed Dirichlet-Neumann boundary condition in a bounded and smooth domain, respectively. Introducing suitable energies and using some observability inequalities, they proved an exponential stability of the solution in both cases under suitable assumptions. Kirane and Said-Houari [13] studied the following initial-boundary problem

$$\begin{cases} u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)ds & \\ \quad +\mu_1 u_t(x,t) + \mu_2 u_t(x,t-\tau) = 0, & (x,t) \in \Omega \times (0,\infty),\\ u_t(x,t-\tau) = f_0(x,t-\tau), & (x,t) \in \Omega \times (0,\tau),\\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), & x \in \Omega,\\ u(x,t) = 0, & (x,t) \in \partial\Omega \times [0,\infty), \end{cases} \tag{1.4}$$

where $\mu_1,\ \mu_2$ are positive constants, $\tau > 0$ represents the time delay. They proved the existence of a unique weak solution for $\mu_2 \leq \mu_1$ relying on the Faedo-Galerkin approximations and some energy estimates. Provided that $g : \mathbb{R}^+ \to \mathbb{R}^+$ is a C^1 function satisfying $g(0) > 0$ and (1.2), and there exists a positive non-increasing differentiable function $\zeta(t)$ such that

$$g'(t) \leq -\zeta(t)g(t) \text{ for } t \geq 0 \text{ and } \int_0^{+\infty} \zeta(t)dt = +\infty, \tag{1.5}$$

by establishing suitable Lyapunov functionals, they also obtained the corresponding exponential stability for $\mu_2 < \mu_1$ and for $\mu_2 = \mu_1$, respectively. Subsequently, Dai and Yang [6] proved an existence result of problem (1.4) without restrictions of $\mu_1,\ \mu_2 > 0$ and $\mu_2 \leq \mu_1$. Making full use of the viscoelasticity term controls the delay term, they also proved an energy decay result for problem (1.4) in the case $\mu_1 = 0$ provided that $g : \mathbb{R}^+ \to \mathbb{R}^+$ is a C^1 function satisfying $g(0) > 0$ and (1.2), and there exists a positive constant ζ such that

$$g(t) \leq -\zeta g(t) \text{ for } t > 0. \tag{1.6}$$

Liu [14] generalized the results obtained by Kirane and Said-Houari [13]. That is, by the similar method in [13], they established a general energy decay result for problem (1.4) with $\tau(t)$ instead of τ. In the absence of the source term $b|u|^{p-2}u$ and the time delay is constant in problem (1.1), Wu [21] proved an energy decay by the similar method in [13], and generalized the results to the time-varying delay in [23]. There are many papers concerning with the stability of viscoelastic equations with time delay, the interested readers may refer to [2, 7, 18] and the reference therein. However, the relaxation function g are mainly limited to satisfying among the three conditions, which are (1.5), (1.6) and that $g : \mathbb{R}^+ \to \mathbb{R}^+$ is a differentiable function satisfying $g(0) > 0$ and (1.2), and there exists a positive function $G \in C^1(\mathbb{R}^+)$ and G is linear or strictly increasing and strictly convex C^2 function on $(0, r]$, $r < 1$, with $G(0) = G'(0) = 0$, such that $g'(t) \leq -G(g(t))$ for $t > 0$. Until recently, Chellaoua and Boukhatem [3] generalized the previous conditions that the relaxation function g satisfied, specifically investigated the following second-order abstract viscoelastic equation in Hilbert spaces

$$u_{tt} + Au - \int_0^{\infty} g(s)Bu(t-s)ds + \mu_1 u_t(t) + \mu_2 u_t(t-\tau) = 0,$$

where $A : D(A) \to H$ and $B : D(B) \to H$ are a self-adjoint linear positive operator with domains $D(A) \subset D(B) \subset H$ such that the embeddings are dense and compact. They established an explicit and general decay results of the energy solution by introducing a suitable Lyapunov functional and some properties of the convex functions under the condition $(\mathbf{H_1})$. Chellaoua and Boukhatem also addressed the stability results for the following second-order abstract viscoelastic equation in Hilbert spaces with time-varying delay in [4]

$$u_{tt} + Au - \int_0^{t} g(t-s)Bu(s)ds + \mu_1 u_t(t) + \mu_2 u_t(t-\tau(t)) = 0$$

under the condition $(\mathbf{H_1})$. It is worth pointing out that Mustafa [16] first proposed the condition $(\mathbf{H_1})$ to study the decay rates for the following equation

$$u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)ds = 0.$$

After that, many authors popularized the method used by Mustafa in [16]. The readers may see the references [1, 8, 9, 15, 17] to get more details.

Motivated by the above works, we are committed to considering the stability of problem (1.1) when the relaxation function g satisfies the condition $(\mathbf{H_1})$. To the best of our knowledge, there is no decay result for problem (1.1) when the relaxation functions satisfy $(\mathbf{H_1})$, although Wu [22] has investigated problem (1.1) with the constant time delay τ and proved the blow-up result with nonpositive and positive initial energy. With minimal conditions on the relaxation function g, the general and optimal energy decay rates of problem (1.1) are established in Theorem 2. Our proof is based on the multiplier method and the similar arguments in [4, 16] but it is different from the previous presentation since the presence of Δu_{tt} and the external force source $b|u|^{p-2}u$. Note that the external force generally promotes the blow-up of the solution.

The outline of this paper is as follows: In Section 2, we give some preliminary lemmas. Section 3 is used to present the energy decay (see Theorem 2) and its proof.

2 Preliminaries

Throughout this paper, we denote by $\|\cdot\|_p$ and $\|\nabla\cdot\|_2$ the norm on $L^p(\Omega)$ with $1 \le p \le \infty$ and $H_0^1(\Omega)$, respectively. Let λ_1 be the first eigenvalue of

$$-\Delta\psi = \lambda\psi, \ x \in \Omega$$

with $\psi = 0, \ x \in \partial\Omega$. The symbol c_s is the optimal embedding constant of $H_0^1(\Omega) \hookrightarrow L^p(\Omega)$.

Motivated by Nicaise and Pignotti [19, 20], let us introduce the new variable $z(x,\kappa,t) = u_t(x, t-\tau(t)\kappa)$ for $x \in \Omega, \ \kappa \in (0,1)$, then problem (1.1) is equivalent to

$$\begin{cases} |u_t|^\rho u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t g(t-s)\Delta u(s)ds & \\ \quad +\mu_1 u_t(x,t) + \mu_2 z(x,1,t) = b|u|^{p-2}u, & (x,t) \in \Omega \times (0,\infty), \\ \tau(t) z_t(x,\kappa,t) + (1-\kappa\tau'(t)) z_\kappa(x,\kappa,t) = 0, & (x,t) \in \Omega \times (0,\infty), \ \kappa \in (0,1), \\ z(x,0,t) = u_t(x,t), & (x,t) \in \Omega \times (0,\infty), \\ z(x,\kappa,0) = f_0(x, -\tau(0)\kappa), & x \in \Omega, \\ u_t(x, t-\tau(0)) = f_0(x, t-\tau(0)), & (x,t) \in \Omega \times (0, \tau(0)), \\ u(x,0) = u_0(x), \ u_t(x,0) = u_1(x), & x \in \Omega, \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times [0,\infty). \end{cases} \tag{2.1}$$

For the completeness of results, in what follows, we state the existence of the solution without proof. In fact, the proof is easy by following Nicaise and Pignotti [19, 20].

Theorem 1. *Let* $(\mathbf{H_1}) - (\mathbf{H_4})$ *hold. Assume that* $u_0,\ u_1 \in H_0^1(\Omega)$ *and* $f_0 \in L^2(\Omega \times (0,1))$*, then there exists a unique solution* (u,z) *of problem* (2.1) *satisfying* $u, u_t \in C([0,T]; H_0^1(\Omega)),\ z \in C([0,T]; L^2(\Omega \times (0,1)))$*, for* $T > 0$.

Remark 1. For $|\mu_2| = \frac{2(1-d)}{2-d}\mu_1$, the above existence theorem still hold. However, the stability of the energy only is given under $|\mu_2| < \frac{2(1-d)}{2-d}\mu_1$.

Define the energy functional of problem (2.1) as follows

$$\begin{aligned} E(t) = & \frac{1}{\rho+2}\left\|u_t\right\|_{\rho+2}^{\rho+2} + \frac{1}{2}\Big(1 - \int_0^t g(s)ds\Big)\left\|\nabla u\right\|_2^2 + \frac{1}{2}(g \circ \nabla u)(t) \\ & + \frac{1}{2}\left\|\nabla u_t\right\|_2^2 + \frac{\xi}{2}\tau(t)\int_\Omega\int_0^1 z^2(x,\kappa,t)d\kappa dx - \frac{b}{p}\left\|u\right\|_p^p, \end{aligned} \tag{2.2}$$

where ξ satisfies $\frac{|\mu_2|}{1-d} \le \xi \le 2\mu_1 - |\mu_2|$, $(g \circ u)(t) = \int_0^t g(t-s)\left\|u(s) - u(t)\right\|_2^2 ds$.

Lemma 1. *The total energy* $E(t)$ *is a non-increasing function and*

$$\begin{aligned} E'(t) & \le -\omega(\|u_t\|_2^2 + \|z(x,1,t)\|_2^2) + \frac{1}{2}(g' \circ \nabla u)(t) - \frac{1}{2}g(t)\left\|\nabla u\right\|_2^2 \\ & \le -\omega(\|u_t\|_2^2 + \|z(x,1,t)\|_2^2) \le 0 \quad \text{for all } t \ge 0, \end{aligned} \tag{2.3}$$

where $\omega = \min\big\{ -\frac{|\mu_2|}{2} + \mu_1 - \frac{\xi}{2}, -\frac{|\mu_2|}{2} + \frac{\xi(1-d)}{2}\big\} \ge 0.$

Proof. This proof is similar to [4, Lemma 2.5]. For convenience, let us give our proof. Multiplying $(2.1)_1$ by u_t, and then integrating over Ω, we get

$$\begin{aligned} & \frac{d}{dt}\Big[\frac{1}{\rho+2}\left\|u_t\right\|^{\rho+2} + \frac{1}{2}\Big(1 - \int_0^t g(s)ds\Big)\left\|\nabla u\right\|_2^2 + \left\|\nabla u_t\right\|_2^2 + (g\circ\nabla u)(t) - \frac{b}{p}\left\|u\right\|_p^p\Big] \\ & - \frac{1}{2}(g' \circ \nabla u)(t) + \frac{1}{2}g(t)\left\|\nabla u(t)\right\|_2^2 + \mu_1\left\|u_t\right\|_2^2 + \mu_2 z(x,1,t)u_t = 0. \end{aligned} \tag{2.4}$$

Here we have used

$$\begin{aligned} & \int_0^t g(t-s)\int_\Omega \nabla u(s)\nabla u_t(t)dxds = -\frac{1}{2}\int_0^t g(t-s)\left(\frac{d}{dt}\left\|\nabla u(s) - \nabla u(t)\right\|_2^2\right)ds \\ & + \frac{1}{2}\int_0^t g(s)\left(\frac{d}{dt}\left\|\nabla u(t)\right\|_2^2\right)ds = -\frac{1}{2}\frac{d}{dt}(g\circ\nabla u)(t) \\ & + \frac{1}{2}(g'\circ\nabla u)(t) + \frac{1}{2}\frac{d}{dt}\int_0^t g(s)\left\|\nabla u(t)\right\|_2^2 ds - \frac{1}{2}g(t)\left\|\nabla u(t)\right\|_2^2. \end{aligned}$$

Multiplying $(2.1)_2$ by $\xi z(x,\kappa,t)$, integrating over Ω, and then integrating over $(0,1)$ with respect to κ, one obtains

$$\frac{\tau(t)\xi}{2}\int_0^1 \frac{\partial}{\partial t}\left\|z(x,\kappa,t)\right\|_2^2 d\kappa + \frac{(1-\kappa\tau'(t))\xi}{2}\int_0^1 \frac{\partial}{\partial\kappa}\left\|z(x,\kappa,t)\right\|_2^2 d\kappa = 0.$$

As a consequence, we have

$$\frac{d}{dt}\Big(\frac{\tau(t)\xi}{2}\int_0^1 \frac{\partial}{\partial t}\left\|z(x,\kappa,t)\right\|_2^2 d\kappa\Big) = \frac{\xi}{2}\left[\left\|u_t\right\|_2^2 - (1-\tau'(t))\left\|z(x,1,t)\right\|_2^2\right]. \tag{2.5}$$

Combining (2.4) with (2.5), and using Cauchy's inequality and ($\mathbf{H_3}$), it follows that

$$E'(t) = \frac{1}{2}(g' \circ \nabla u)(t) - \frac{1}{2}g(t) \|\nabla u(t)\|_2^2 + \Big(\frac{|\mu_2|}{2} - \mu_1 + \frac{\xi}{2}\Big) \|u_t\|_2^2 + \Big(\frac{|\mu_2|}{2} - \frac{\xi(1-d)}{2}\Big) \|z(x,1,t)\|_2^2 .$$

□

Lemma 2. *If u is a solution for problem (2.1) and $E(0) < E_1 = \frac{p-2}{2p}\sigma_1^2$, $l\|\nabla u_0\|_2^2 < \sigma_1^2$, here $\sigma_1 = b^{-\frac{1}{p-2}} B_1^{-\frac{p}{p-2}}$, $B_1 = c_s^p l^{-\frac{p}{2}}$, then there exists a positive constant σ_2 satisfying $0 < \sigma_2 < \sigma_1$ such that*

$$l\|\nabla u\|_2^2 + (g \circ \nabla u)(t) \le \sigma_2^2 \quad \text{for all } t \ge 0. \tag{2.6}$$

Proof. Taking the combination of equations (2.2) and (1.2) with the embedding $H_0^1(\Omega) \hookrightarrow L^p(\Omega)$, one has

$$E(t) \ge F\Big(\sqrt{l\|\nabla u\|_2^2 + (g \circ \nabla u)(t)}\Big), \tag{2.7}$$

where $F(x) = \frac{1}{2}x^2 - \frac{bB_1^p}{p}x^p$ for $x > 0$. We know that F is strictly increasing in $(0, \sigma_1)$, strictly decreasing in (σ_1, ∞), and F has a maximum at σ_1 with the maximum value E_1. Since $E(0) < E_1$, there exists a $\sigma_2 < \sigma_1$ such that $F(\sigma_2) = E(0)$. Set $\sigma_0 := \sqrt{l\|\nabla u_0\|_2^2}$, recall (2.7), then $F(\sigma_0) \le E(0) = F(\sigma_2)$, which implies $\sigma_0 \le \sigma_2$ due to the given condition $\sigma_0^2 < \sigma_1^2$. To complete the proof of (2.6), we suppose by contradiction that for some $t^0 > 0$,

$$\sigma(t^0) = \sqrt{l\|\nabla u(t_0)\|_2^2 + (g \circ \nabla u)(t_0)} > \sigma_2.$$

The continuity of $\sqrt{l\|\nabla u\|_2^2 + (g \circ \nabla u)(t)}$ illustrates that we may choose t^0 such that $\sigma_1 > \sigma(t^0) > \sigma_2$, then we have $E(0) = F(\sigma_2) < F(\sigma(t^0)) \le E(t^0)$. This is a contradiction because of Lemma 1. □

Lemma 3. *Under all the conditions of Lemma 2, there exists a positive constant $\mathcal{D}$ such that for all $t \ge 0$,*

$$\|u\|_p^p \le \mathcal{D}E(t) \le \mathcal{D}E(0), \tag{2.8}$$

$$\frac{1}{\rho+2}\|u_t\|_{\rho+2}^{\rho+2} + \frac{1}{2}\Big(1 - \int_0^t g(s)ds\Big)\|\nabla u\|_2^2 + \frac{1}{2}(g \circ \nabla u)(t) + \frac{1}{2}\|\nabla u_t\|_2^2 + \frac{\xi\tau(t)}{2}\int_\Omega\int_0^1 z^2(x,\kappa,t)d\kappa dx \le \mathcal{D}E(t) \le \mathcal{D}E(0). \tag{2.9}$$

Proof. Using the embedding $H_0^1(\Omega) \hookrightarrow L^p(\Omega)$, (2.2) and (2.6), we have

$$\begin{aligned}
\frac{b}{p}\|u\|_p^p &\le \frac{bB_1^p}{p}\Big(l\|\nabla u\|_2^2+(g\circ\nabla u)(t)\Big)^{\frac{p-2}{2}}\left[l\|\nabla u\|_2^2+(g\circ\nabla u)(t)\right]\\
&\le \frac{2bB_1^p}{p}\Big(l\|\nabla u\|_2^2+(g\circ\nabla u)(t)\Big)^{\frac{p-2}{2}}\Big(E(t)+\frac{b}{p}\|u\|_p^p\Big)\\
&\le \frac{2bB_1^p}{p}\sigma_2^{p-2}\Big(E(t)+\frac{b}{p}\|u\|_p^p\Big),
\end{aligned}$$

which yields (2.8) with $\mathcal{D}=\frac{2pB_1^p\sigma_2^{p-2}}{p-2bB_1^p\sigma_2^{p-2}}>0$. One has (2.9) by combining (2.8) with (2.2). □

Lemma 4 [Lemma 4.1 in [1]]. *For $u\in H_0^1(\Omega)$, we have for all $t\ge 0$,*

$$\int_\Omega\Big(\int_0^t g(t-s)(\nabla u(s)-\nabla u(t))ds\Big)^2dx\le C_\alpha(h_\alpha\circ\nabla u)(t) \tag{2.10}$$

where, for any $0<\alpha<1$,

$$C_\alpha=\int_0^\infty\frac{g^2(s)}{\alpha g(s)-g'(s)}ds \text{ and } h_\alpha(t)=\alpha g(t)-g'(t). \tag{2.11}$$

Let us follow from the proof of Lemma 4.1 in [1], we have in fact

$$\int_\Omega\Big(\int_0^t g(t-s)(u(s)-u(t))ds\Big)^2dx\le C_\alpha(h_\alpha\circ u)(t). \tag{2.12}$$

Lemma 5 [Lemma 2.2 in [4]]. *There exist positive constants γ and t_1 such that*

$$g'(t)\le-\gamma g(t)\quad \text{for } t\in[0,t_1]. \tag{2.13}$$

Lemma 6. *Let u be a solution of problem (2.1), then the functional*

$$I_1(t)=\frac{1}{\rho+1}\int_\Omega|u_t|^\rho u_tudx+\int_\Omega\nabla u_t\nabla udx, \tag{2.14}$$

satisfies, for $\varepsilon>0$ and for all $t\ge0$,

$$\begin{aligned}
I_1'(t)&\le\frac{1}{\rho+1}\|u_t\|_{\rho+2}^{\rho+2}-\Big[l-\Big(1+\frac{\mu_1}{\lambda_1}+\frac{\mu_2}{\lambda_1}\Big)\varepsilon\Big]\|\nabla u\|_2^2 \qquad (2.15)\\
&+\frac{1}{4\varepsilon}C_\alpha(h_\alpha\circ\nabla u)(t)+\frac{|\mu_2|}{4\varepsilon}\|z(x,1,t)\|_2^2+b\|u\|_p^p+\|\nabla u_t\|_2^2+\frac{\mu_1}{4\varepsilon}\|u_t\|_2^2.
\end{aligned}$$

Proof. Multiplying $(2.1)_1$ by u, integrating on x over Ω, and then using integration by parts, we give

$$\begin{aligned}
\int_\Omega|u_t|^\rho u_{tt}udx+\int_\Omega\nabla u_{tt}\nabla udx&=-\Big(1-\int_0^t g(s)ds\Big)\|\nabla u\|_2^2+\int_\Omega\nabla u(t)\int_0^t g(t-s)\\
&\times\nabla(u(s)-u(t))ds-\int_\Omega\mu_1u_t(x,t)udx-\int_\Omega\mu_2z(x,1,t)udx+b\|u\|_p^p. \qquad (2.16)
\end{aligned}$$

Differentiating (2.14) on t, and using (2.16), one has

$$\begin{aligned} I_1'(t) &= \frac{1}{\rho+1}\|u_t\|_{\rho+2}^{\rho+2} + \int_\Omega |u_t|^\rho u_{tt} u dx + \int_\Omega \nabla u_{tt} \nabla u dx + \|\nabla u_t\|_2^2 \\ &= -\Big(1 - \int_0^t g(s)ds\Big)\|\nabla u\|_2^2 + \int_\Omega \nabla u(t) \int_0^t g(t-s)\nabla(u(s)-u(t))ds \\ &+ \frac{1}{\rho+1}\|u_t\|_{\rho+2}^{\rho+2} - \int_\Omega \mu_1 u_t(x,t) u dx - \int_\Omega \mu_2 z(x,1,t) u dx + b\|u\|_p^p + \|\nabla u_t\|_2^2 . \end{aligned} \tag{2.17}$$

Applying Cauchy's inequality with $\varepsilon > 0$ and $\lambda_1 \|u\|_2^2 \le \|\nabla u\|_2^2$, it follows that

$$\begin{aligned} -\int_\Omega \mu_1 u_t(x,t) u dx &\le \frac{\mu_1}{4\varepsilon}\|u_t\|_2^2 + \mu_1\varepsilon\|u\|_2^2 \le \frac{\mu_1}{4\varepsilon}\|u_t\|_2^2 + \frac{\mu_1\varepsilon}{\lambda_1}\|\nabla u\|_2^2 , \\ -\int_\Omega \mu_2 z(x,1,t) u dx &\le \frac{|\mu_2|}{4\varepsilon}\|z(x,1,t)\|_2^2 + \frac{|\mu_2|\varepsilon}{\lambda_1}\|\nabla u\|_2^2 . \end{aligned} \tag{2.18}$$

It follows from Cauchy's inequality with $\varepsilon > 0$ and (2.10) that

$$\int_\Omega \nabla u(t) \int_0^t g(t-s)(\nabla u(s) - \nabla u(t))ds \le \varepsilon\|\nabla u\|_2^2 + \frac{1}{4\varepsilon} C_\alpha (h \circ \nabla u)(t). \tag{2.19}$$

Inserting (2.18)–(2.19) into (2.17), we obtain (2.15). □

Lemma 7. *Under all the conditions of Lemma* 2, *let u be a solution of problem* (2.1), *then the functional*

$$I_2(t) = \int_\Omega \Big(\Delta u_t - \frac{1}{\rho+1}|u_t|^\rho u_t\Big) \int_0^t g(t-s)(u(t)-u(s))dsdx, \tag{2.20}$$

satisfies, for $\delta > 0$ and for all $t \ge 0$,

$$\begin{aligned} I_2'(t) &\le B_1\|\nabla u\|_2^2 + B_2(h_\alpha \circ \nabla u)(t) + \Big[B_3 - \int_0^t g(s)ds\Big]\|\nabla u_t\|_2^2 \\ &+ \delta\|z(x,1,t)\|_2^2 - \int_0^t g(s)ds \cdot \frac{1}{\rho+1}\|u_t\|_{\rho+2}^{\rho+2} , \end{aligned} \tag{2.21}$$

here B_1, B_2 and B_3 are positive constants depending on δ shown in (2.25).

Proof. Differentiating (2.20) on t, and using $(2.1)_1$ and integration by parts yield

$$\begin{aligned} I_2'(t) &= \int_\Omega \nabla u(t) \int_0^t g(t-s)\nabla(u(t)-u(s))ds \\ &- \int_\Omega \int_0^t g(t-s)\nabla u(s)ds \int_0^t g(t-s)\nabla(u(t)-u(s))dsdx \\ &+ \int_\Omega \mu_1 u_t(x,t) \int_0^t g(t-s)(u(t)-u(s))dsdx \end{aligned}$$

$$
\begin{aligned}
&+\int_\Omega \mu_2 z(x,1,t)\int_0^t g(t-s)(u(t)-u(s))dsdx - b\int_\Omega |u|^{p-2}u\\
&\times\int_0^t g(t-s)(u(t)-u(s))dsdx - \int_\Omega \nabla u_t\int_0^t g_t(t-s)\nabla(u(t)-u(s))dsdx\\
&-\int_\Omega \frac{1}{\rho+1}|u_t|^\rho u_t\int_0^t g_t(t-s)(u(t)-u(s))dsdx\\
&-\int_0^t g(s)ds\cdot\|\nabla u_t\|_2^2 - \int_0^t g(s)ds\cdot\frac{1}{\rho+1}\|u_t\|_{\rho+2}^{\rho+2} \qquad (2.22)\\
&= J_1 + J_2 + \cdots + J_8 + J_9.
\end{aligned}
$$

It is direct from Cauchy's inequality with $\delta > 0$ and (2.10) that

$$
\begin{aligned}
J_1 &\le \delta\|\nabla u\|_2^2 + \frac{1}{4\delta}C_\alpha(h_\alpha\circ\nabla u)(t),\\
J_2 &\le 2\delta\int_\Omega\Big(\int_0^t g(t-s)|\nabla u(s)-\nabla u(t)|ds\Big)^2dx + 2\delta(1-l)^2\|\nabla u(t)\|_2^2\\
&\quad + \frac{1}{4\delta}\int_\Omega\Big(\int_0^t g(t-s)|\nabla u(s)-\nabla u(t)|ds\Big)^2dx\\
&\le \Big(2\delta+\frac{1}{4\delta}\Big)C_\alpha(h_\alpha\circ\nabla u)(t) + 2\delta(1-l)^2\|\nabla u\|_2^2. \qquad (2.23)
\end{aligned}
$$

Cauchy's inequality with $\delta > 0$ and (2.12) yield

$$
\begin{aligned}
J_3 &\le \delta\|u_t\|_2^2 + \frac{\mu_1^2}{4\delta}C_\alpha(h_\alpha\circ u)(t) \le \frac{\delta}{\lambda_1}\|\nabla u_t\|_2^2 + \frac{\mu_1^2}{4\delta\lambda_1}C_\alpha(h_\alpha\circ\nabla u)(t),\\
J_4 &\le \delta\|z(x,1,t)\|_2^2 + \frac{\mu_2^2}{4\delta\lambda_1}C_\alpha(h_\alpha\circ\nabla u)(t).
\end{aligned}
$$

It follows from Cauchy's inequality with $\delta > 0$, (2.12), the embedding $H_0^1(\Omega)\hookrightarrow L^{2(p-1)}(\Omega)$ and (2.9) that

$$
\begin{aligned}
J_5 &\le b\delta\|u\|_{2(p-1)}^{2(p-1)} + \frac{b}{4\delta}\int_\Omega\Big(\int_0^t g(t-s)(u(t)-u(s))ds\Big)^2dx\\
&\le b\delta\|u\|_{2(p-1)}^{2(p-1)} + \frac{b}{4\delta}C_\alpha(h_\alpha\circ u)(t)\\
&\le b\delta c_s^{2(p-1)}\Big(\frac{2\mathcal{D}}{l}E(0)\Big)^{p-2}\|\nabla u\|_2^2 + \frac{b}{4\delta\lambda_1}C_\alpha(h_\alpha\circ\nabla u)(t).
\end{aligned}
$$

Recalling the definition of $g'(t)$ in (2.11), and using Cauchy's inequality with $\delta > 0$, (2.10) and Hölder's inequality, one obtains

$$
\begin{aligned}
J_6 &= -\int_\Omega \nabla u_t\int_0^t \alpha g(t-s)\nabla(u(t)-u(s))dsdx\\
&\quad + \int_\Omega \nabla u_t\int_0^t h_\alpha(t-s)\nabla(u(t)-u(s))dsdx\\
&\le \delta\|\nabla u_t\|_2^2 + \frac{\alpha^2}{4\delta}\int_\Omega\Big(\int_0^t g(t-s)\nabla(u(t)-u(s))ds\Big)^2dx
\end{aligned}
$$

$$
\begin{aligned}
&+\delta\left\|\nabla u_t\right\|_2^2+\frac{1}{4\delta}\int_\Omega\Big(\int_0^t h_\alpha(t-s)\nabla(u(t)-u(s))ds\Big)^2dx\le 2\delta\left\|\nabla u_t\right\|_2^2\\
&+\frac{\alpha^2}{4\delta}C_\alpha(h_\alpha\circ\nabla u)(t)+\frac{1}{4\delta}\int_0^t h_\alpha(s)ds\int_0^t h_\alpha(t-s)\left\|\nabla(u(t)-u(s))\right\|_2^2ds\\
\le{}& 2\delta\left\|\nabla u_t\right\|_2^2+\Big(\frac{\alpha^2}{4\delta}C_\alpha+\frac{\alpha(1-l)+g(0)}{4\delta}\Big)(h_\alpha\circ\nabla u)(t).
\end{aligned}
$$

Similarly, we get

$$
\begin{aligned}
J_7={}&-\int_\Omega\frac{1}{\rho+1}|u_t|^\rho u_t\int_0^t\alpha g(t-s)(u(t)-u(s))dsdx\\
&+\int_\Omega\frac{1}{\rho+1}|u_t|^\rho u_t\int_0^t h_\alpha(t-s)(u(t)-u(s))dsdx\\
\le{}&\frac{\delta}{\rho+1}\left\|u_t\right\|_{2(\rho+1)}^{2(\rho+1)}+\frac{\alpha^2}{4(\rho+1)\delta}\int_\Omega\Big(\int_0^t g(t-s)(u(t)-u(s))ds\Big)^2dx\\
&+\frac{\delta}{\rho+1}\left\|u_t\right\|_{2(\rho+1)}^{2(\rho+1)}+\frac{1}{4(\rho+1)\delta}\int_\Omega\Big(\int_0^t h_\alpha(t-s)(u(t)-u(s))ds\Big)^2dx\\
\le{}&\frac{2\delta}{\rho+1}\left\|u_t\right\|_{2(\rho+1)}^{2(\rho+1)}+\frac{\alpha^2}{4(\rho+1)\delta}C_\alpha(h_\alpha\circ u)(t)+\frac{1}{4(\rho+1)\delta}\int_0^t h_\alpha(s)ds\\
&\times\int_0^t h_\alpha(t-s)\left\|u(t)-u(s)\right\|_2^2ds\le\frac{2\delta}{\rho+1}c_s^{2(\rho+1)}\Big(2\mathcal{D}E(0)\Big)^{\frac{\rho}{2}}\left\|\nabla u_t\right\|_2^2\\
&+\Big(\frac{\alpha^2}{4(\rho+1)\delta}C_\alpha+\frac{\alpha(1-l)+g(0)}{4(\rho+1)\delta}\Big)\frac{1}{\lambda_1}(h_\alpha\circ\nabla u)(t).
\end{aligned}
\tag{2.24}
$$

Inserting (2.23)–(2.24) into (2.22), one has

$$
\begin{aligned}
I_2'(t)\le{}& B_1\left\|\nabla u\right\|_2^2+B_2(h_\alpha\circ\nabla u)(t)+\Big[B_3-\int_0^t g(s)ds\Big]\left\|\nabla u_t\right\|_2^2\\
&+\delta\left\|z(x,1,t)\right\|_2^2-\int_0^t g(s)ds\cdot\frac{1}{\rho+1}\left\|u_t\right\|_{\rho+2}^{\rho+2},
\end{aligned}
$$

with

$$
\begin{cases}
B_1=\delta+2\delta(1-l)^2+b\delta c_s^{2(p-1)}\Big(\frac{2\mathcal{D}}{l}E(0)\Big)^{p-2};\\
B_2=\Big[\frac{1}{2\delta}+2\delta+\frac{\mu_1^2}{4\delta\lambda_1}+\frac{\mu_2^2}{4\delta\lambda_1}+\frac{b}{4\delta\lambda_1}+\frac{\alpha^2}{4\delta}+\frac{\alpha^2}{4(\rho+1)\delta\lambda_1}\Big]C_\alpha\\
\qquad+\frac{\alpha(1-l)+g(0)}{4\delta}+\frac{\alpha(1-l)+g(0)}{4(\rho+1)\delta\lambda_1};\\
B_3=\frac{\delta}{\lambda_1}+2\delta+\frac{2\delta}{\rho+1}c_s^{2(\rho+1)}\Big(2\mathcal{D}E(0)\Big)^{\frac{\rho}{2}}.
\end{cases}
\tag{2.25}
$$

□

Lemma 8 [Lemma 2.8 in [4]]. *The functional*

$$
I_3(t)=\int_0^1 e^{-2\tau(t)\kappa}\left\|z(x,\kappa,t)\right\|_2^2d\kappa
$$

satisfies for all $t \geq 0$,

$$I_3'(t) = -2I_3(t) + \frac{1}{\tau_0}\|u_t\|_2^2 - \frac{(1-d)e^{-2\tau_1}}{\tau_1}\|z(x,1,t)\|_2^2. \tag{2.26}$$

Lemma 9 [Lemma 3.4 in [16]]. *The functional*

$$I_4(t) = \int_0^t f(t-s)\|\nabla u(s)\|_2^2\,ds \tag{2.27}$$

satisfies for all $t \geq 0$,

$$I_4'(t) \leq 3(1-l)\|\nabla u\|_2^2 - \frac{1}{2}(g \circ \nabla u)(t), \tag{2.28}$$

where $f(t) = \int_t^\infty g(s)ds$.

3 Stability results

In this section, we will present and prove the decay results of the energy functional $E(t)$ based on the lemmas in Section 2. To begin with, we define a functional

$$L(t) = ME(t) + \sum_{i=1}^{3} N_i I_i(t), \tag{3.1}$$

where $M,\ N_1,\ N_2,\ N_3$ are positive constants. The following lemma is shown to illustrate that $L(t)$ is equivalent to $E(t)$.

Lemma 10. *Under all the conditions of Lemma* 2, *assume that* M *is enough large, then there exist two positive constants* β_1 *and* β_2 *such that*

$$\beta_1 E(t) \leq L(t) \leq \beta_2 E(t).$$

Proof. Recalling the definition of $I_1(t)$ in (2.14), using Young's inequality and Cauchy's inequality, and then applying the embedding $H_0^1(\Omega) \hookrightarrow L^{\rho+2}(\Omega)$ and (2.9), it is not hard to give

$$\begin{aligned} |I_1(t)| &\leq \frac{1}{\rho+2}\|u_t\|_{\rho+2}^{\rho+2} + \frac{1}{(\rho+1)(\rho+2)}\|u\|_{\rho+2}^{\rho+2} + \frac{1}{2}\|\nabla u_t\|_2^2 + \frac{1}{2}\|\nabla u\|_2^2 \\ &\leq \frac{1}{\rho+2}\|u_t\|_{\rho+2}^{\rho+2} + \Big[\frac{c_s^{\rho+2}}{(\rho+1)(\rho+2)}\Big(\frac{2\mathcal{D}}{l}E(0)\Big)^{\frac{\rho}{2}} + \frac{1}{2}\Big]\|\nabla u\|_2^2 + \frac{1}{2}\|\nabla u_t\|_2^2. \end{aligned}$$

Recalling the definition of $I_2(t)$ in (2.20), using integration by parts, Cauchy's inequality, Young's inequality and Hölder's inequality, we give

$$\begin{aligned} |I_2(t)| &\leq \frac{1}{2}\|\nabla u_t\|_2^2 + \frac{1}{2}\int_\Omega \Big(\int_0^t g(t-s)(\nabla u(t) - \nabla u(s))ds\Big)^2 dx \\ &\quad + \frac{1}{\rho+2}\|u_t\|_{\rho+2}^{\rho+2} + \frac{1}{(\rho+1)(\rho+2)}\int_\Omega \Big(\int_0^t g(t-s)(u(t)-u(s))ds\Big)^{\rho+2} dx \\ &\leq \frac{1}{2}\|\nabla u_t\|_2^2 + \frac{1-l}{2}(g \circ \nabla u)(t) + \frac{1}{\rho+2}\|u_t\|_{\rho+2}^{\rho+2} \\ &\quad + \frac{1}{(\rho+1)(\rho+2)}(1-l)^{\rho+1}c_s^{\rho+2}\Big(\frac{2\mathcal{D}}{l}E(0)\Big)^{\frac{\rho}{2}}(g \circ \nabla u)(t), \end{aligned}$$

where we have used

$$\begin{aligned}
&\int_\Omega \Big(\int_0^t g(t-s)(u(t)-u(s))ds\Big)^{\rho+2}dx\\
&\quad\le \int_\Omega \Big(\int_0^t (g(t-s))^{\frac{\rho+1}{\rho+2}}(g(t-s))^{\frac{1}{\rho+2}}(u(t)-u(s))ds\Big)^{\rho+2}dx\\
&\quad\le \Big(\int_0^t g(s)ds\Big)^{\rho+1}\int_0^t g(t-s)\left\|u(t)-u(s))\right\|_{\rho+2}^{\rho+2}ds\\
&\quad\le (1-l)^{\rho+1}c_s^{\rho+2}\int_0^t g(t-s)\left\|\nabla u(t)-\nabla u(s))\right\|_2^{\rho+2}ds\\
&\quad\le (1-l)^{\rho+1}c_s^{\rho+2}\Big(\frac{2\mathcal{D}}{l}E(0)\Big)^{\frac{\rho}{2}}(g\circ\nabla u)(t).
\end{aligned}$$

Therefore, it follows from (2.9) that

$$|L(t)-ME(t)| = \left|\sum_{i=1}^{3}N_iI_i(t)\right| \le CE(t),$$

where C is some positive constant. □

Lemma 11. *Under all the conditions of Lemma* 2, *the functional* $L(t)$ *defined in* (3.1) *satisfies, for* $t\ge t_1$

$$\begin{aligned}
L'(t) &\le -C_1\left\|u_t\right\|_{\rho+2}^{\rho+2} - C_2\left\|\nabla u_t\right\|_2^2 - 4(1-l)\left\|\nabla u\right\|_2^2\\
&\quad + N_1b\left\|u\right\|_p^p + \frac{1}{4}(g\circ\nabla u)(t) - 2N_3\int_0^1 e^{-2\tau(t)\kappa}\left\|z(x,\kappa,t)\right\|_2^2d\kappa. \qquad (3.2)
\end{aligned}$$

where $C_1,\ C_2$ *are positive constants given in* (3.4).

Proof. Taking the combination of (2.3), (2.15) and (2.21) with (2.26), recalling (2.11), and applying $g_1=\int_0^{t_1}g(s)ds\le\int_0^t g(s)ds$ for $t\ge t_1$, one has

$$\begin{aligned}
L'(t) &\le -M\omega(\left\|u_t\right\|_2^2+\left\|z(x,1,t)\right\|_2^2) + \frac{M}{2}(g'\circ\nabla u)(t) - \frac{M}{2}g(t)\left\|\nabla u\right\|_2^2 \qquad (3.3)\\
&+ \frac{N_1}{\rho+1}\left\|u_t\right\|_{\rho+2}^{\rho+2} - N_1\Big[l-\Big(1+\frac{\mu_1}{\lambda_1}+\frac{\mu_2}{\lambda_1}\Big)\varepsilon\Big]\left\|\nabla u\right\|_2^2 + \frac{N_1}{4\varepsilon}C_\alpha(h_\alpha\circ\nabla u)(t)\\
&+ \frac{N_1|\mu_2|}{4\varepsilon}\left\|z(x,1,t)\right\|_2^2 + N_1b\left\|u\right\|_p^p + N_1\left\|\nabla u_t\right\|_2^2 + N_1\frac{\mu_1}{4\varepsilon}\left\|u_t\right\|_2^2\\
&+ N_2B_1\left\|\nabla u\right\|_2^2 + N_2B_2(h_\alpha\circ\nabla u)(t) + N_2\Big[B_3-\int_0^t g(s)ds\Big]\left\|\nabla u_t\right\|_2^2\\
&+ N_2\delta\left\|z(x,1,t)\right\|_2^2 - N_2\int_0^t g(s)ds\cdot\frac{1}{\rho+1}\left\|u_t\right\|_{\rho+2}^{\rho+2}\\
&- 2N_3I_3(t) + \frac{N_3}{\tau_0}\left\|u_t\right\|_2^2 - \frac{N_3(1-d)e^{-2\tau_1}}{\tau_1}\left\|z(x,1,t)\right\|_2^2\\
&\le -C_1\left\|u_t\right\|_{\rho+2}^{\rho+2} - C_2\left\|\nabla u_t\right\|_2^2 - C_3\left\|\nabla u\right\|_2^2 - C_4\left\|z(x,1,t)\right\|_2^2 - C_5(h_\alpha\circ\nabla u)(t)
\end{aligned}$$

$$+ N_1 b \|u\|_p^p + \frac{\alpha M}{2}(g \circ \nabla u)(t) - C_6 \|u_t\|_2^2 - 2N_3 \int_0^1 e^{-2\tau(t)\kappa} \|z(x,\kappa,t)\|_2^2 \, d\kappa$$

with

$$\begin{cases} C_1 = N_2 g_1 \cdot \frac{1}{\rho+1} - \frac{N_1}{\rho+1}; \\ C_2 = N_2 \Big[g_1 - \Big(\frac{\delta}{\lambda_1} + 2\delta + \frac{2\delta}{\rho+1} c_s^{2(\rho+1)} \Big(2\mathcal{D}E(0) \Big)^{\frac{\rho}{2}} \Big) \Big] - N_1; \\ C_3 = N_1 \Big[l - \Big(1 + \frac{\mu_1}{\lambda_1} + \frac{|\mu_2|}{\lambda_1} \Big) \varepsilon \Big] - N_2 \Big[\delta + 2\delta(1-l)^2 + b\delta c_s^{2(p-1)} \Big(\frac{2\mathcal{D}}{l} E(0) \Big)^{p-2} \Big]; \\ C_4 = \omega M + \frac{N_3(1-d)e^{-2\tau_1}}{\tau_1} - N_1 \frac{|\mu_2|}{4\varepsilon} - N_2 \delta; \\ C_5 = \frac{M}{2} - N_1 \frac{1}{4\varepsilon} C_\alpha - N_2 \Big[\Big(\frac{1}{2\delta} + 2\delta + \frac{\mu_1^2}{4\delta\lambda_1} + \frac{\mu_2^2}{4\delta\lambda_1} + \frac{b}{4\delta\lambda_1} + \frac{\alpha^2}{4\delta} + \frac{\alpha^2}{4(\rho+1)\delta\lambda_1} \Big) C_\alpha \\ \qquad + \frac{\alpha(1-l)+g(0)}{4\delta} + \frac{\alpha(1-l)+g(0)}{4(\rho+1)\delta\lambda_1} \Big]; \\ C_6 = \omega M - \frac{N_3}{\tau_0} - N_1 \frac{\mu_1}{4\varepsilon}, \end{cases} \tag{3.4}$$

where we have used the values of B_1, B_2 and B_3 defined in (2.25).

Next, we choose δ such that

$$\delta < \Bigg\{ \frac{l g_1}{16 \Big[1 + 2(1-l)^2 + b c_s^{2(p-1)} \Big(\frac{2\mathcal{D}}{l} E(0) \Big)^{p-2} \Big]}, \frac{l g_1}{1024(1-l)^2}, \\ \frac{5}{8} g_1 \Big/ \Big(\frac{1}{\lambda_1} + 2 + \frac{2}{\rho+1} c_s^{2(\rho+1)} \Big(2\mathcal{D}E(0) \Big)^{\frac{\rho}{2}} \Big) \Bigg\}.$$

Let us choose $N_1 = \frac{3}{8} g_1 N_2$, then

$$C_1 = N_2 g_1 \cdot \frac{1}{\rho+1} - \frac{3}{8} g_1 N_2 \frac{1}{\rho+1} = \frac{5}{8} g_1 N_2 \frac{1}{\rho+1} > 0, \quad C_2 > 0.$$

Let us fix $\varepsilon = \frac{3l}{4} \frac{1}{1 + \frac{\mu_1}{\lambda_1} + \frac{|\mu_2|}{\lambda_1}}$, then

$$C_3 = \frac{N_1 l}{4} - N_2 \Big[\delta + 2\delta(1-l)^2 + b\delta c_s^{2(p-1)} \Big(\frac{2\mathcal{D}}{l} E(0) \Big)^{p-2} \Big] > \frac{l}{32} g_1 N_2 > 0.$$

By taking $N_2 = \frac{1}{8\delta(1-l)}$, we get $C_3 > \frac{l}{32} g_1 N_2 = \frac{l g_1}{256\delta(1-l)} > 4(1-l)$.

Since $g'(s) \le 0$, one has $\frac{\alpha g^2(s)}{\alpha g(s) - g'(s)} \le g(s)$, further we get

$$\lim_{\alpha \to 0^+} \alpha C_\alpha = \lim_{\alpha \to 0^+} \int_0^\infty \frac{\alpha g^2(s)}{\alpha g(s) - g'(s)} ds = 0.$$

Thus, there exists $0 < \alpha_0 < 1$ so that if $\alpha < \alpha_0$, then

$$\alpha C_\alpha < \frac{1}{8 \Big[N_2 \Big(\frac{1}{2\delta} + 2\delta + \frac{\mu_1^2}{4\delta\lambda_1} + \frac{\mu_2^2}{4\delta\lambda_1} + \frac{b}{4\delta\lambda_1} + \frac{\alpha^2}{4\delta} + \frac{\alpha^2}{4(\rho+1)\delta\lambda_1} \Big) + N_1 \frac{1}{4\varepsilon} \Big]}.$$

Let us choose M sufficiently large such that for $\alpha = \frac{1}{2M}$,

$$C_5 = \frac{M}{4} - N_2 \Big[\frac{\alpha(1-l)+g(0)}{4\delta} + \frac{\alpha(1-l)+g(0)}{4(\rho+1)\delta\lambda_1} \Big] > 0,$$

$$C_4 = \omega M + \frac{N_3(1-d)e^{-2\tau_1}}{\tau_1} - N_1\frac{|\mu_2|}{4\varepsilon} - N_2\delta > 0, C_6 = \omega M - \frac{N_3}{\tau_0} - N_1\frac{\mu_1}{4\varepsilon} > 0.$$

Here we have used $\omega > 0$ given in Lemma 1 based on the condition $(\mathbf{H_4})$.

Based on the above discussion, one has from (3.3)

$$\begin{aligned} L'(t) \le& -C_1 \|u_t\|_{\rho+2}^{\rho+2} - C_2 \|\nabla u_t\|_2^2 - 4(1-l)\|\nabla u\|_2^2 \\ &+ N_1 b \|u\|_p^p + \frac{1}{4}(g\circ\nabla u)(t) - 2N_3\int_0^1 e^{-2\tau(t)\kappa}\|z(x,\kappa,t)\|_2^2\, d\kappa. \end{aligned}$$

□

Now, we give the following stability results.

Theorem 2. *Let* $(\mathbf{H_1})-(\mathbf{H_4})$ *hold, and* $E(0) < E_1$, $l\|\nabla u_0\|_2^2 < \sigma_1^2$, *then there exist positive constants* k_1, k_2, k_3 *and* k_4 *such that the solution of problem* (1.1) *satisfies for all* $t \ge t_1$,

$$E(t) \le \begin{cases} k_1 e^{-k_2\int_{t_1}^t \zeta(s)ds} & \text{for } G \text{ is linear;} \\ k_4 G_1^{-1}\big(k_3\int_{t_1}^t \zeta(s)ds\big) & \text{for } G \text{ is nonlinear,} \end{cases}$$

where E_1 *and* σ_1 *are shown in Lemma 2,* $G_1(t) = \int_t^r \frac{1}{sG'(s)}ds$ *is strictly decreasing and convex in* $(0,r]$ *with* $\lim_{t\to 0} G_1(t) = +\infty$.

Proof. Using (2.13) and (2.3), one has, for $t \ge t_1$,

$$\begin{aligned} &\int_0^{t_1} g(s)\|\nabla u(t) - \nabla u(t-s)\|_2^2\, ds \\ &\quad \le -\frac{1}{\gamma}\int_0^{t_1} g'(s)\|\nabla u(t) - \nabla u(t-s)\|_2^2\, ds \le -cE'(t). \end{aligned} \tag{3.5}$$

Here c is used to denote a generic positive constant throughout this proof. Define a functional $F(t)$ that is obviously equivalent to $E(t)$ as follows

$$F(t) = L(t) + cE(t),$$

then based on (3.2), (2.2), (3.5), for some $m > 0$ and for any $t \ge t_1$, we have

$$\begin{aligned} F'(t) \le& -C_1 \|u_t\|_{\rho+2}^{\rho+2} - C_2\|\nabla u_t\|_2^2 - 4(1-l)\|\nabla u\|_2^2 \\ &+ N_1 b\|u\|_p^p + \frac{1}{4}(g\circ\nabla u)(t) - 2N_3\int_0^1 e^{-2\tau(t)\kappa}\|z(x,\kappa,t)\|_2^2\, d\kappa + cE'(t) \\ \le& -mE(t) - \Big(\frac{bc}{p} - N_1 b\Big)\|u\|_p^p + c(g\circ\nabla u)(t) + cE'(t) \\ \le& -mE(t) + c\int_{t_1}^t g(s)\|\nabla u(t) - \nabla u(t-s)\|_2^2\, ds, \end{aligned} \tag{3.6}$$

where we have chosen N_1 so small that $bc/p - N_1 b > 0$.

In what follows, we will discuss in two cases.

Case 1: G is linear. Multiplying (3.6) by $\zeta(t)$, using $(\mathbf{H_1})$ and (2.3), one gives

$$\zeta(t)F'(t) \leq -m\zeta(t)E(t) + c\zeta(t)\int_{t_1}^{t} g(s)\left\|\nabla u(t)-\nabla u(t-s)\right\|_2^2 ds$$
$$\leq -m\zeta(t)E(t) - c\int_{t_1}^{t} g'(s)\left\|\nabla u(t)-\nabla u(t-s)\right\|_2^2 ds \leq -m\zeta(t)E(t) - cE'(t),$$

which implies

$$(\zeta(t)F(t)+cE(t))' \leq -m\zeta(t)E(t) \quad \text{for } t \geq t_1.$$

Integrating the above inequality over (t_1,t), and using the fact that $\zeta(t)F(t)+cE(t)$ is equivalent to $E(t)$, one has

$$E(t) \leq k_1 e^{-k_2\int_{t_1}^{t}\zeta(s)ds} \quad \text{for } t \geq t_1,$$

where k_1 and k_2 are constants Case 2: G is nonlinear. Define a functional $H(t) = L(t) + I_4(t)$. Taking the combination of Lemma 10 and the non-negativity of $E(t)$ obtained by Lemma 3 with the definition of $I_4(t)$ in (2.27), it is not difficult to get the non-negativity of $H(t)$. It follows from (3.2) and (2.28) that for some $m_1 > 0$ and $t \geq t_1$,

$$H'(t) = L'(t) + I_4'(t) \leq -C_1\left\|u_t\right\|_{\rho+2}^{\rho+2} - C_2\left\|\nabla u_t\right\|_2^2 - (1-l)\left\|\nabla u\right\|_2^2$$
$$+ N_1 b\left\|u\right\|_p^p - \frac{1}{4}(g\circ\nabla u)(t) - 2N_3\int_0^1 e^{-2\tau(t)\kappa}\left\|z(x,\kappa,t)\right\|_2^2 d\kappa \leq -m_1E(t).$$

Integrating the above inequality over (t_1,t) yields

$$m_1\int_{t_1}^{t} E(s)ds \leq H(t_1) - H(t) \leq H(t_1),$$

which implies

$$\int_0^{\infty} E(s)ds < +\infty.$$

Define

$$\lambda(t) = p\int_{t_1}^{t}\left\|\nabla u(t)-\nabla u(t-s)\right\|_2^2 ds,$$

by using (2.9), then we give

$$\lambda(t) \leq 2p\int_0^t\Big(\left\|\nabla u(t)\right\|_2^2 + \left\|\nabla u(t-s)\right\|_2^2\Big)ds \leq \frac{8p\mathcal{D}}{l}\int_0^t\Big(E(t)+E(t-s)\Big)ds$$
$$\leq \frac{16p\mathcal{D}}{l}\int_0^t E(t-s)ds = \frac{16p\mathcal{D}}{l}\int_0^t E(s)ds \leq \int_0^{\infty} E(s)ds < +\infty.$$

Thus, we can choose p so small that for $t \geq t_1$,

$$\lambda(t) < 1. \tag{3.7}$$

It is direct that

$$G(\theta z) \leq \theta G(z) \quad \text{for } 0 \leq \theta \leq 1 \text{ and } z \in (0, r], \tag{3.8}$$

since G is strictly convex on $(0, r]$ and $G(0) = 0$. Based on (1.3), (3.7), (3.8) and Jensen's inequality, one gives

$$\begin{aligned}
I(t) &= \frac{1}{p\lambda(t)} \int_{t_1}^{t} \lambda(t)(-g'(s))p \left\| \nabla u(t) - \nabla u(t-s) \right\|_2^2 ds \geq \frac{1}{p\lambda(t)} \int_{t_1}^{t} \lambda(t)\zeta(s)G(g(s)) \\
&\times p \left\| \nabla u(t) - \nabla u(t-s) \right\|_2^2 ds \geq \frac{\zeta(t)}{p\lambda(t)} \int_{t_1}^{t} \overline{G}(\lambda(t)g(s))p \left\| \nabla u(t) - \nabla u(t-s) \right\|_2^2 ds \\
&\geq \frac{\zeta(t)}{p} \overline{G}\Big(p \int_{t_1}^{t} g(s) \left\| \nabla u(t) - \nabla u(t-s) \right\|_2^2 ds\Big),
\end{aligned}$$

which yields

$$\int_{t_1}^{t} g(s) \left\| \nabla u(t) - \nabla u(t-s) \right\|_2^2 ds \leq \frac{1}{p} \overline{G}^{-1}\Big(\frac{pI(t)}{\zeta(t)}\Big),$$

where G has an extension $\overline{G}$ which is a strictly increasing and strictly convex C^2 function on $(0, +\infty)$ [4, Remark 2.1]. Therefore, (3.6) becomes

$$F'(t) \leq -mE(t) + \frac{c}{p} \overline{G}^{-1}\Big(\frac{pI(t)}{\zeta(t)}\Big). \tag{3.9}$$

Let us define the functional

$$F_1(t) = \overline{G}'\big(r_1 E(t)/E(0)\big)F(t) + E(t)$$

with $0 < r_1 < r$, then F_1 is equivalent to E and

$$\begin{aligned}
F_1'(t) &= \frac{r_1 E'(t)}{E(0)} \overline{G}''\Big(\frac{r_1 E(t)}{E(0)}\Big)F(t) + \overline{G}'\Big(\frac{r_1 E(t)}{E(0)}\Big)F'(t) + E'(t) \\
&\leq -mE(t)\overline{G}'\Big(\frac{r_1 E(t)}{E(0)}\Big) + \frac{c}{p} \overline{G}^{-1}\Big(\frac{pI(t)}{\zeta(t)}\Big)\overline{G}'\Big(\frac{r_1 E(t)}{E(0)}\Big) + E'(t)
\end{aligned} \tag{3.10}$$

by using (3.9), (2.3), $G' > 0$ and $G'' > 0$. Let $\overline{G}^*$ be the convex conjugate of G in the sense of Young, which is given by

$$\overline{G}^*(s) = s(\overline{G}')^{-1}(s) - \overline{G}\Big[(\overline{G}')^{-1}(s)\Big] \tag{3.11}$$

and it satisfies the following Young's inequality

$$AB \leq \overline{G}^*(A) + \overline{G}(B). \tag{3.12}$$

Choosing

$$A = \overline{G}'\big(r_1 E(t)/E(0)\big) \text{ and } B = \overline{G}^{-1}\big(pI(t)/\zeta(t)\big),$$

then using (3.12), (3.11) and the non-negativity of $\overline{G}$, (3.10) becomes

$$F_1'(t) \leq -mE(t)\overline{G}'\Big(\frac{r_1 E(t)}{E(0)}\Big) + \frac{c}{p} \overline{G}^{-1}\Big(\frac{pI(t)}{\zeta(t)}\Big)\overline{G}'\Big(\frac{r_1 E(t)}{E(0)}\Big) + E'(t)$$

$$
\begin{aligned}
&\le -mE(t)\overline{G}'\Big(\frac{r_1E(t)}{E(0)}\Big) + \frac{c}{p}\overline{G}^*\Big(\overline{G}'\Big(\frac{r_1E(t)}{E(0)}\Big)\Big) + c\frac{I(t)}{\zeta(t)} + E'(t)\\
&\le -mE(t)\overline{G}'\Big(\frac{r_1E(t)}{E(0)}\Big)+\frac{c}{p}\frac{r_1E(t)}{E(0)}\overline{G}'\Big(\frac{r_1E(t)}{E(0)}\Big)-\overline{G}\Big(\frac{r_1E(t)}{E(0)}\Big)+c\frac{I(t)}{\zeta(t)}+E'(t)\\
&\le -mE(t)\overline{G}'\Big(\frac{r_1E(t)}{E(0)}\Big) + \frac{c}{p}\frac{r_1E(t)}{E(0)}\overline{G}'\Big(\frac{r_1E(t)}{E(0)}\Big) + c\frac{I(t)}{\zeta(t)} + E'(t). \qquad (3.13)
\end{aligned}
$$

Note that (2.3) implies

$$
I(t) \le \int_{t_1}^{t} -g'(s)\,\|\nabla u(t) - \nabla u(t-s)\|_2^2\, ds \le -2E'(t),
$$

then one has

$$
\zeta(t)F_1'(t) \le -m\zeta(t)E(t)G'\Big(\frac{r_1E(t)}{E(0)}\Big) + \frac{c}{p}\zeta(t)\frac{r_1E(t)}{E(0)}G'\Big(\frac{r_1E(t)}{E(0)}\Big) - cE'(t)
$$

multiplying (3.13) by $\zeta(t)$ and using the fact

$$
\overline{G}'\big(r_1E(t)/E(0)\big) = G'\big(r_1E(t)/E(0)\big).
$$

Define the functional $F_2(t) = \zeta(t)F_1(t) + cE(t)$ which is equivalent to $E(t)$, which means

$$
\gamma_1F_2(t) \le E(t) \le \gamma_2F_2(t) \qquad (3.14)
$$

for some γ_1 and γ_2. Under a suitable choice of r_1 and for a positive constant k, we have

$$
F_2'(t) \le -k\zeta(t)\frac{E(t)}{E(0)}G'\Big(\frac{r_1E(t)}{E(0)}\Big) = -k\zeta(t)G_2\Big(\frac{E(t)}{E(0)}\Big) \qquad (3.15)
$$

with $G_2(t) = tG'(r_1t)$. Obviously, G_2 and G_2' are positive in $(0,1]$ since

$$
G_2'(t) = G'(r_1t) + r_1t^2G''(r_1t)
$$

and the convexity of G in $(0,r]$. Inequalities (3.15) and (3.14) imply

$$
\Big(\frac{\gamma_1F_2(t)}{E(0)}\Big)' \le -k\zeta(t)\frac{\gamma_1}{E(0)}G_2\Big(\frac{E(t)}{E(0)}\Big) \le -k_3\zeta(t)G_2\Big(\frac{\gamma_1F_2(t)}{E(0)}\Big) \qquad (3.16)
$$

with $k_3 = k\frac{\gamma_1}{E(0)}$. Setting $R(t) = \frac{\gamma_1F_2(t)}{E(0)}$, and then integrating (3.16) over (t_1,t), one has

$$
\int_{t_1}^{t} -\frac{R'(s)}{G_2(R(s))}ds \ge \int_{t_1}^{t} k_3\zeta(s)ds.
$$

Since $r_1R(t_1) < r$, we have

$$
G_1(r_1R(t)) = \int_{r_1R(t)}^{r_1R(t_1)} \frac{1}{sG'(s)}ds \ge k_3\int_{t_1}^{t}\zeta(s)ds.
$$

It is noted that G_1 is strictly decreasing function on $(0, r]$ and $\lim_{t\to 0} G_1(t) = +\infty$ in Theorem 2, then $R(t) \leq \frac{1}{r_1} G_1^{-1}\Big(k_3 \int_{t_1}^{t} \zeta(s)ds\Big)$. Since $R(t)$ is equivalent to $E(t)$, further one obtains

$$E(t) \leq k_4 G_1^{-1}\Big(k_3 \int_{t_1}^{t} \zeta(s)ds\Big)$$

with $k_4 = \frac{1}{r_1}$. This completes the proof of this theorem. □

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