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Optimization Problems for a Thermoelastic Frictional Contact Problem

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Abstract. In the present paper, we analyze and study the control of a static thermoelastic contact problem. We consider a model which describes a frictional contact problem between a thermoelastic body and a deformable heat conductor obstacle. We derive a variational formulation of the model which is in the form of a coupled system of the quasi-variational inequality of elliptic type for the displacement and the nonlinear variational equation for the temperature. Then, under a smallness assumption, we prove the existence of a unique weak solution to the problem. Moreover, we establish the dependence of the solution with respect to the data and prove a convergence result. Finally, we introduce an optimization problem related to the contact model for which we prove the existence of a minimizer and provide a convergence result.

Keywords: thermo-elastic material, frictional contact, variational coupled system, convergence results, optimization problem.

AMS Subject Classification: 35J87; 47J20; 49J40; 74F15; 74G30; 74M10; 74M15; 74S05.

1 Introduction

Currently, the study of contact problems involving thermo-elastic materials remains an active research area. Due to the intrinsic coupling between mechanical and thermal energy, these materials has attracted the attention of industry and engineering researchers. For this reason, a considerable effort has been made in its modelling and numerical simulations of contact problems, and the literature concerning this topic is rather extensive. For instance, we can see [2,3,4,6,7,8] for general thermoelastic models and their analysis, [9,14,15,16,21,22] for the mathematical treatment of optimal control for a system governed by variational equations and inequalities. Moreover, we refer to [1,11,12,13,17,19,20] and more recently [5,18] for some comprehensive references on analysis optimal control problems arising from contact models.

The aim of this paper is to deal with a model describing the static process of frictional contact between a thermo-elastic body and a deformable foundation. After proving the unique weak solvability of the contact problem, as well as a convergence result of the solution with respect to the data, we consider an optimization problem related to our contact problem, for which we provide under some smallness conditions, the existence of a minimizer and a convergence result.

The paper is organized as follows. In Section 2, we introduce the thermoelastic frictional contact model, we list the assumptions on the data and derive its variational formulation, which is given as a coupled system for the displacement and the temperature fields. In Section 3, we state and prove the main existence and uniqueness result, Theorem 1. In Section 4, we prove the continuous dependence of the weak solution on the set of constraints with respect to the data and prove a convergence result, Theorem 2. Finally, in Section 5, we introduce a class of optimization problems related to the contact model and provide their solvability, Theorem 3. In addition, we give two examples of optimization problems that illustrate our results.

2 A frictional thermoelastic contact problem

The physical setting of our contact problem is described as follows: we consider a thermoelastic body occupying, in its reference configuration, a bounded domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ with a sufficiently regular boundary Γ. The boundary Γ is partitioned into four disjoint measurable parts Γ_1, Γ_2, Γ_3 and Γ_4, such that $meas(\Gamma_1) > 0$. The body is clamped on Γ_1 and is subjected to a given volume force f_0 and heat source q_0 in Ω. Moreover, it is acted upon by a given surface traction f_2 on Γ_2 and a null variation of temperature on $\Gamma_1 \cup \Gamma_2$. Finally, the body could come in frictional contact with two obstacles over Γ_3 and Γ_4.

To derive the mathematical model describing the previous physical setting, let $u(x)$, $\sigma(x)$, $\theta(x)$ and $q_T(x)$ represent the displacement field, the stress tensor field, the temperature field, and the heat flux vector field, respectively. In what follows, to simplify the notation, we do not indicate explicitly the dependence of various functions on the spatial variable $x \in \Omega \cup \Gamma$, that is, we write, for example, σ instead of $\sigma(x)$. Moreover, the summation convention over repeated indices is used and the index that follows a comma indicates the partial derivative with respect to the corresponding component of the independent variable, e.g. $u_{i,j} = \partial u_i/\partial x_j$.

Let $\mathbb{S}^d$ be the space of second-order symmetric tensors on $\mathbb{R}^d$, or equivalently, the space of symmetric matrices of order d. We define the inner products and the associated norms on $\mathbb{R}^d$ and $\mathbb{S}^d$ as follows

$$u \cdot v = u_i v_i, \quad \|v\| = (v \cdot v)^{\frac{1}{2}}, \quad \forall\, u, v \in \mathbb{R}^d,$$
$$\sigma \cdot \tau = \sigma_{ij}\tau_{ij}, \quad \|\tau\| = (\tau \cdot \tau)^{\frac{1}{2}}, \quad \forall\, \sigma, \tau \in \mathbb{S}^d.$$

Let ν denotes the unit outward normal to the boundary Γ. Then, the normal and tangential components of the displacement vector $v \in \mathbb{R}^d$ and the stress tensor $\sigma \in \mathbb{S}^d$ on Γ are given by

$$v_\nu = v \cdot \nu, \quad v_\tau = v - v_\nu \nu; \qquad \sigma_\nu = (\sigma\nu) \cdot \nu, \quad \sigma_\tau = \sigma\nu - \sigma_\nu \nu.$$

Under these notations, the frictional thermoelastic contact problem can be formulated as follows.

Problem $[\mathcal{P}]$. Find a displacement $u : \Omega \to \mathbb{R}^d$ and a temperature $\theta : \Omega \to \mathbb{R}$ such that

$$\sigma = \mathcal{F}\varepsilon(u) - \mathcal{M}\theta \quad \text{in} \quad \Omega, \tag{2.1}$$
$$q_T = -\mathcal{K}\nabla\theta \quad \text{in} \quad \Omega, \tag{2.2}$$
$$\operatorname{Div}\sigma + f_0 = 0 \quad \text{in} \quad \Omega, \quad \operatorname{div} q_T = q_0 \quad \text{in} \quad \Omega, \tag{2.3}$$
$$u = 0 \quad \text{on} \quad \Gamma_1, \quad \sigma\nu = f_2 \quad \text{on} \quad \Gamma_2, \quad \theta = 0 \quad \text{on} \quad \Gamma_1 \cup \Gamma_2, \tag{2.4}$$
$$\left.\begin{array}{l} \sigma_\nu = -S, \\ \|\sigma_\tau\| \le S, \\ \|\sigma_\tau\| < S \;\Rightarrow\; u_\tau = 0, \\ \sigma_\tau = -S\,\frac{u_\tau}{\|u_\tau\|} \;\Rightarrow\; \exists\lambda > 0 \quad u_\tau = -\lambda\sigma_\tau \end{array}\right\} \quad \text{on} \quad \Gamma_3, \tag{2.5}$$
$$q_T \cdot \nu = k_T(u_\nu - g)\,\varphi_L(\theta - \theta_F) \quad \text{on} \quad \Gamma_3, \tag{2.6}$$
$$\left.\begin{array}{l} \sigma_\nu = -p_\nu(u_\nu - g)\,h_\nu(\theta - \theta_F), \\ \|\sigma_\tau\| \le p_\tau(u_\nu - g)\,h_\tau(\theta - \theta_F), \\ \|\sigma_\tau\| < p_\tau(u_\nu - g)\,h_\tau(\theta - \theta_F) \;\text{ if }\; u_\tau = 0, \\ \sigma_\tau = -p_\tau(u_\nu - g)\,h_\tau(\theta - \theta_F)\frac{u_\tau}{\|u_\tau\|} \;\text{ if }\; u_\tau \neq 0 \end{array}\right\} \quad \text{on} \quad \Gamma_4, \tag{2.7}$$
$$-q_T \cdot \nu \le 0, \quad (\theta - \theta_F) \le 0, \quad (q_T \cdot \nu)\,(\theta - \theta_F) = 0 \quad \text{on} \quad \Gamma_4. \tag{2.8}$$

Equations (2.1)–(2.2) represent the thermo-elastic constitutive law of the material in which $\mathcal{F} = (f_{ijkl})$, $\mathcal{M} = (m_{ij})$ and $\mathcal{K} = (k_{ij})$ are the elastic, the thermal expansion and thermal conductivity tensors, where $\varepsilon(u) = (\varepsilon_{ij}) = (\frac{1}{2}(u_{i,j} + u_{j,i}))$ is the linearized strain tensor. Equations (2.3) are the equilibrium equations for the stress and the heat flux fields where Div and div denote the divergence operator, respectively for tensor and vector valued functions. The relations (2.4) are the mechanical and thermal boundary conditions. Conditions (2.5) represents Tresca's contact model, i.e., a nonpositive normal stress

$-S$ is imposed on given contact surface and the tangential stress is bounded by prescribed friction bound S, so if such a limit is not attained, sliding does not occur. Equation (2.6) represents the heat flow between a body and heat conductor foundation, where g is the gap function between the body and the foundation on the contact interface Γ_3 or Γ_4, $\theta_F \in \mathbb{R}_+^*$ is the temperature of the foundation, k_T is the coefficient of heat exchange between it and the body, and φ_L is the truncation function defined for a given large constant $L > 0$ by

$$\varphi_L(s) = \begin{cases} s & \text{if} \quad |s| \le L, \\ L\,\frac{s}{|s|} & \text{if} \quad |s| > L. \end{cases} \tag{2.9}$$

Note that φ_L is L-bounded and 1-Lipschitz continuous function. Relations (2.7) describes the normal compliance contact condition coupled with Coulomb's friction law over Γ_4, where p_ν is a prescribed nonnegative function depending on the relative temperature $\theta-\theta_F$ and vanishing for negative arguments, p_τ is a given function depending on $u_\nu-g$, and $h_\tau \ge 0$ is the coefficient of friction which depends on $\theta - \theta_F$. Finally, Equation (2.8) denotes temperature dependent Signorini's law. It means that the heat flux is assumed to be unilateral from the foundation to the body, and then the body temperature does not exceed the foundation's temperature θ_F on the contact parts.

In order to derive a weak formulation of Problem $(\mathcal{P})$, we introduce the following spaces

$$H = L^2(\Omega)^d, \quad H_1 = H^1(\Omega)^d, \quad \mathcal{H} = \big\{\tau = (\tau_{ij})\,;\ \tau_{ij} = \tau_{ji} \in L^2(\Omega)\big\},$$

which are real Hilbert spaces for the following inner products and their associated norms

$$(u,v)_H = \int_\Omega u_i v_i \, dx, \quad (u,v)_{H_1} = (u,v)_H + (\varepsilon(u),\varepsilon(v))_{\mathcal{H}}, \quad (\sigma,\tau)_{\mathcal{H}} = \int_\Omega \sigma_{ij}\tau_{ij}\, dx.$$

Then, we consider the sets of admissible displacements and temperatures, defined by

$$\begin{aligned} &V = \{v \in H_1\,, \quad v = 0 \ \text{ on } \ \Gamma_1\}, \quad Q = \{\xi \in H^1(\Omega)\,, \quad \xi = 0 \ \text{ on } \ \Gamma_1 \cup \Gamma_2\}, \\ &W = \{\xi \in Q\,, \quad \xi \le \theta_F \ \text{ on } \ \Gamma_4\}. \end{aligned}$$

Over the spaces V and Q, we consider the following inner products and associated norms

$$\begin{aligned} (u,v)_V &= (\varepsilon(u),\varepsilon(v))_{\mathcal{H}}\,, \qquad \|u\|_V = (u,u)_V^{1/2}, \quad \forall\, u,\, v \in V, \\ (\theta,\xi)_Q &= (\nabla\theta,\nabla\xi)_H\,, \quad \|\theta\|_Q = (\theta,\theta)_Q^{1/2}, \quad \forall\, \theta,\, \xi \in Q. \end{aligned}$$

Since Γ_1 is on non-zero measure, the following Korn and Friedrichs-Poincaré inequalities hold, for some positive constants c_k and c_p, depending only on Ω and Γ_1 such that

$$\|\varepsilon(v)\|_{\mathcal{H}} \ge c_k\, \|v\|_{H_1}, \quad \forall\, v \in V, \tag{2.10}$$

$$\|\nabla\xi\|_H \ge c_p\, \|\xi\|_{H^1(\Omega)}, \quad \forall\, \xi \in Q. \tag{2.11}$$

Hence $(V, \|\cdot\|_V)$ and $(Q, \|\cdot\|_Q)$ are real Hilbert spaces. Moreover, by the Sobolev trace theorem, there exists positive constants c_1 and c_2 depending only on Ω, Γ_1, $\Gamma_c = \Gamma_3$ or Γ_4 such that

$$\|v\|_{L^2(\Gamma_c)^d} \leq c_1 \|v\|_V, \quad \forall v \in V, \tag{2.12}$$

$$\|\xi\|_{L^2(\Gamma_c)} \leq c_2 \|\xi\|_Q, \quad \forall \xi \in Q. \tag{2.13}$$

To study of the mechanical problem $(\mathcal{P})$, we need the following hypotheses

$(\mathcal{H}_1)$ The elasticity tensor $\mathcal{F} = (f_{ijkl}) : \Omega \times \mathbb{S}^d \to \mathbb{S}^d$ and the thermal conductivity tensor $\mathcal{K} = (k_{ij}) : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ satisfy the following usual properties $f_{ijkl} = f_{jikl} = f_{lkij} \in L^\infty(\Omega)$, $k_{ij} = k_{ji} \in L^\infty(\Omega)$ and there exists a nonnegative constants $m_\mathcal{F}$ and $m_\mathcal{K}$ such that

$$f_{ijkl}(x)\,\tau_{kl}\,\tau_{ij} \geq m_\mathcal{F}\,\|\tau\|^2, \quad \forall \tau = (\tau_{ij}) \in \mathbb{S}^d \text{ a.e. } x \in \Omega,$$

$$k_{ij}(x)\,\zeta_i\,\zeta_j \geq m_\mathcal{K}\,\|\zeta\|^2, \quad \forall \zeta = (\zeta_i) \in \mathbb{R}^d \text{ a.e. } x \in \Omega.$$

Let $M_\mathcal{F} = \sup\limits_{i,j,k,l} \|f_{ijkl}\|_{L^\infty(\Omega)}$, $M_\mathcal{K} = \sup\limits_{i,j} \|k_{ij}\|_{L^\infty(\Omega)}$ be the norms of $\mathcal{F}$ and $\mathcal{K}$.

$(\mathcal{H}_2)$ The thermal expansion tensor $\mathcal{M} = (m_{ij}) : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ satisfies $m_{ij} = m_{ji} \in L^\infty(\Omega)$. Let $\|\mathcal{M}\| = \sup\limits_{i,j} \|m_{ij}\|_{L^\infty(\Omega)}$ be the norm of the thermal expansion tensor $\mathcal{M}$.

$(\mathcal{H}_3)$ The compliance function $p_r : \Gamma_4 \times \mathbb{R} \to \mathbb{R}_+$ $(r = \nu, \tau)$ satisfies

(a) $\exists\, M_{p_r} > 0$ such that $|p_r(x,u)| \leq M_{p_r}$ for all $u \in \mathbb{R}$, a.e. $x \in \Gamma_4$,

(b) $\forall\, u \in \mathbb{R}$, $x \mapsto p_r(x,u)$ is measurable on Γ_4 and is zero for all $u \leq 0$,

(c) $\exists\, L_{p_r} > 0, \forall\, u_1, u_2 \in \mathbb{R}$, $|p_r(x,u_1) - p_r(x,u_2)| \leq L_{p_r}|u_1 - u_2|$ a.e. $x \in \Gamma_4$.

$(\mathcal{H}_4)$ The function $h_r : \Gamma_4 \times \mathbb{R} \to \mathbb{R}_+$ $(r = \nu, \tau)$ satisfies the properties

(a) $\exists\, M_{h_r} > 0$ such that $|h_r(x,\theta)| \leq M_{h_r}$ for all $\theta \in \mathbb{R}$, a.e. $x \in \Gamma_4$,

(b) $\forall\, \theta \in \mathbb{R}$, $x \mapsto h_r(x,\theta)$ is measurable on Γ_4,

(c) $\exists\, L_{h_r} > 0, \forall\, \theta_1, \theta_2 \in \mathbb{R}$, $|h_r(x,\theta_1) - h_r(x,\theta_2)| \leq L_{h_r}|\theta_1 - \theta_2|$ a.e. $x \in \Gamma_4$.

$(\mathcal{H}_5)$ The thermal conductance $k_T : \Gamma_3 \times \mathbb{R} \to \mathbb{R}_+$ satisfies

(a) $\exists\, M_{k_T} > 0, \forall\, u \in \mathbb{R}$, $|k_T(x,u)| \leq M_{k_T}$ a.e. $x \in \Gamma_3$,

(b) $\forall\, u \in \mathbb{R}$, $x \mapsto k_T(x,u)$ is measurable on Γ_3,

(c) $\exists L_{k_T} > 0, \forall u_1, u_2 \in \mathbb{R}, |k_T(x,u_1) - k_T(x,u_2)| \leq L_{k_T}|u_1 - u_2|$ a.e. $x \in \Gamma_3$.

$(\mathcal{H}_6)$ The body forces, traction and heat source densities satisfy

$$f_0 \in L^2(\Omega)^d,\ f_2 \in L^2(\Gamma_2)^d,\ q_0 \in L^2(\Omega).$$

($\mathcal{H}_7$) The friction bound, the gap function and temperature of the foundation satisfy

$$S \geq 0 \text{ a.e. } x \in \Gamma_3\ , \quad g \geq 0 \text{ a.e. } x \in \Gamma_3 \cup \Gamma_4\ , \quad \theta_F \in \mathbb{R}_+^*,$$
$$S \in L^2(\Gamma_3) \quad \text{and} \quad g \in L^2(\Gamma_3 \cup \Gamma_4).$$

Now, we use the Riesz representation theorem to define $f \in V$ and $q \in Q$ by

$$(f, v)_V = \int_\Omega f_0 \cdot v\, dx + \int_{\Gamma_2} f_2 \cdot v\, da - \int_{\Gamma_3} S \cdot v_\nu\, da, \quad \forall\, v \in V,$$
$$(q, \xi)_Q = \int_\Omega q_0\, \xi\, dx, \quad \forall\, \xi \in Q.$$

We introduce the two mappings $j_S : V \to \mathbb{R}$ and $l : V \times Q \times Q \to \mathbb{R}$ defined by

$$j_S(v) = \int_{\Gamma_3} S\, \|v_\tau\|\, da, \;\; l(u, \theta, \xi) = \int_{\Gamma_3} k_T(u_\nu - g)\, \varphi_L(\theta - \theta_F)\, \xi\, da. \tag{2.14}$$

We also introduce the functionals j_ν, j_τ and j defined on $V \times Q \times V$ as follows

$$j_\nu(u, \theta, v) = \int_{\Gamma_4} p_\nu(u_\nu - g)\, h_\nu(\theta - \theta_F)\, v_\nu\, da, \tag{2.15}$$
$$j_\tau(u, \theta, v) = \int_{\Gamma_4} p_\tau(u_\nu - g)\, h_\tau(\theta - \theta_F)\, \|v_\tau\|\, da,$$
$$j(u, \theta, v) = j_\nu(u, \theta, v) + j_\tau(u, \theta, v). \tag{2.16}$$

Then, we deduce that the variational formulation of Problem ($\mathcal{P}$) is as follows.

Problem [$\mathcal{PV}$]. Find a displacement field $u \in V$ and a temperature field $\theta \in W$ such that

$$\begin{aligned}&(\mathcal{F}\varepsilon(u), \varepsilon(v) - \varepsilon(u))_{\mathcal{H}} - (\mathcal{M}\theta, \varepsilon(v) - \varepsilon(u))_{\mathcal{H}} \\ &\quad + j_S(v) - j_S(u) + j(u, \theta, v) - j(u, \theta, u) \geq (f, v - u)_V, \quad \forall\, v \in V,\end{aligned} \tag{2.17}$$

$$(\mathcal{K}\nabla\theta, \nabla\xi - \nabla\theta)_H + l(u, \theta, \xi - \theta) \geq (q, \xi - \theta)_Q, \quad \forall\, \xi \in W. \tag{2.18}$$

Problem ($\mathcal{PV}$) is formulated in terms of displacement field u and temperature field θ, and once the two fields u and θ are known, the stress tensor σ and the heat flux vector q_T can be deduced by using the equations (2.1) and (2.2). The analysis and the unique solvability of Problem ($\mathcal{PV}$) will be provided in the next section.

3 The unique solvability of Problem ($\mathcal{PV}$)

First, we consider the two following nonnegative constants

$$L_{\mathcal{F}} = \frac{m_{\mathcal{F}}}{\max\left(\frac{1}{2\, c_p}, \frac{c_1 c_2}{2}, c_1^2\right)}, \quad L_{\mathcal{K}} = \frac{m_{\mathcal{K}}}{\max\left(\frac{1}{2\, c_p}, \frac{c_1 c_2}{2}, c_2^2\right)}.$$

Next, the unique solvability of Problem ($\mathcal{PV}$) is provided by the below theorem.

Theorem 1. *Assume the hypotheses* $(\mathcal{H}_1)$–$(\mathcal{H}_7)$ *hold. Then, Problem* $(\mathcal{PV})$ *has at least one solution* $(u, \theta) \in V \times W$. *Moreover, if the following conditions hold*

$$\begin{aligned} &\|\mathcal{M}\| + LL_{k_T} + L_{p_\nu} M_{h_\nu} + L_{h_\nu} M_{p_\nu} + L_{p_\tau} M_{h_\tau} + L_{h_\tau} M_{p_\tau} < L_{\mathcal{F}}, \\ &\|\mathcal{M}\| + LL_{k_T} + M_{k_T} + L_{h_\nu} M_{p_\nu} + L_{h_\tau} M_{p_\tau} < L_{\mathcal{K}}, \end{aligned} \tag{3.1}$$

then, Problem $(\mathcal{PV})$ *has a unique solution.*

The proof of Theorem 1 will be carried out in several steps. First, we introduce the two product spaces $X = V \times Q$ and $Y = Q \times L^2(\Gamma_4)^2 \times L^2(\Gamma_3)$, which are real Hilbert spaces for the following inner products and their associated Euclidian norms $\| \cdot \|_X$ and $\| \cdot \|_Y$.

$$\begin{aligned} (x, y)_X &= (u, v)_V + (\theta, \xi)_Q, \\ (\eta, \zeta)_Y &= (\eta_1, \zeta_1)_Q + \sum_{j=2,3} (\eta_j, \zeta_j)_{L^2(\Gamma_4)} + (\eta_4, \zeta_4)_{L^2(\Gamma_3)}, \end{aligned}$$

for all $x = (u, \theta)$, $y = (v, \xi) \in X$ and $\eta = (\eta_i)_i$, $\zeta = (\zeta_i)_i \in Y$. For a given $\eta = (\eta_i)_i \in Y$, we consider functionals $j^\eta : V \to \mathbb{R}$ and $\ell^\eta : Q \to \mathbb{R}$ defined by

$$j^\eta(v) = j_S(v) + \int_{\Gamma_4} \eta_2 \, v_\nu \, da + \int_{\Gamma_4} \eta_3 \, \|v_\tau\| \, da, \tag{3.2}$$

$$\ell^\eta(\xi) = \int_{\Gamma_3} \eta_4 \, \xi \, da. \tag{3.3}$$

Then, we can now introduce the following intermediate problem.

Problem $[\mathcal{PV}^\eta]$. Given $\eta = (\eta_i)_{i=1,..,4} \in Y$, find $x^\eta = (u^\eta, \theta^\eta) \in U = V \times W$ such that

$$\begin{aligned} (\mathcal{F}\varepsilon(u^\eta), \varepsilon(v) - \varepsilon(u^\eta))_{\mathcal{H}} - (\mathcal{M}\eta_1, \varepsilon(v) - \varepsilon(u^\eta))_{\mathcal{H}} \\ + j^\eta(v) - j^\eta(u^\eta) \geq (f, v - u^\eta)_V, \ \forall \, v \in V, \end{aligned} \tag{3.4}$$

$$(\mathcal{K}\nabla\theta^\eta, \nabla\xi - \nabla\theta^\eta)_H + \ell^\eta(\xi - \theta^\eta) \geq (q, \xi - \theta^\eta)_Q, \ \forall \, \xi \in W. \tag{3.5}$$

To prove the unique solvability of Problem $(\mathcal{PV}^\eta)$, we consider the operator $A : X \to X$, the element $F^\eta \in X$ and the functional $J^\eta : X \to \mathbb{R}$ defined by

$$(Ax, y)_X = (\mathcal{F}\varepsilon(u), \varepsilon(v))_{\mathcal{H}} + (\mathcal{K}\nabla\theta, \nabla\xi)_H, \tag{3.6}$$

$$(F^\eta, y)_X = (f, v)_V + (q, \xi)_Q + (\mathcal{M}\eta_1, \varepsilon(u))_{\mathcal{H}}, \tag{3.7}$$

$$J^\eta(y) = j^\eta(v) + \ell^\eta(\xi), \tag{3.8}$$

where $x = (u, \theta)$, $y = (v, \xi) \in X$. Then, we have the following Lemma

Lemma 1. *For any given* $\eta \in Y$, *we have the following results*

1. The couple $x^\eta = (u^\eta, \theta^\eta)$ *is a solution of Problem* $(\mathcal{PV}^\eta)$ *if and only if*

$$(Ax^\eta, y - x^\eta)_X + J^\eta(y) - J^\eta(x^\eta) \geq (F^\eta, y - x^\eta)_X, \quad \forall \, y = (v, \xi) \in U. \tag{3.9}$$

2. The problem $(\mathcal{PV}^\eta)$ has unique solution $x^\eta = (u^\eta, \theta^\eta) \in U = V \times W$.

3. The mapping $\eta \mapsto (u^\eta, \theta^\eta)$ is Lipschitz continuous on Y.

Proof. Let $x^\eta = (u^\eta, \theta^\eta)$ a solution of Problem $(\mathcal{PV}^\eta)$, we add the two inequalities (3.4) and (3.5), and we use (3.6)–(3.8) to obtain (3.9). Conversely, if $x^\eta = (u^\eta, \theta^\eta) \in U$ satisfies the elliptic inequality (3.9), we take $y = (u^\eta, \xi)$ in (3.9) where ξ is an arbitrary element of W, to obtain (3.5), and for an arbitrary $v \in V$, we take $y = (v, \theta^\eta)$ in the inequality (3.9) to get (3.4), which concludes the first part of Lemma 1.

For the second part of Lemma 1, it follows from the definition (3.6) and the hypothesis $(\mathcal{H}_1)$ that for all $x_1 = (u_1, \theta_1)$ and $x_2 = (u_2, \theta_2)$ of X, we have

$$\begin{aligned}(Ax_1 - Ax_2, x_1 - x_2)_X &= (\mathcal{F}\varepsilon(u_1) - \mathcal{F}\varepsilon(u_2), \varepsilon(u_1) - \varepsilon(u_2))_{\mathcal{H}} \\ &\quad + (\mathcal{K}\nabla\theta_1 - \mathcal{K}\nabla\theta_2, \nabla\theta_1 - \nabla\theta_2)_H \geq m_{\mathcal{F}}\|u_1 - u_2\|_V^2 + m_{\mathcal{K}}\|\theta_1 - \theta_2\|_Q^2 \\ &\geq \underbrace{\min(m_{\mathcal{F}}, m_{\mathcal{K}})}_{=m_A} \|x_1 - x_2\|_X^2. \end{aligned} \tag{3.10}$$

Moreover, for all $x_1 = (u_1, \theta_1)$, $x_2 = (u_2, \theta_2)$ and $y = (v, \xi)$ of X, we have

$$(Ax_1 - Ax_2, y)_X = (\mathcal{F}\varepsilon(u_1) - \mathcal{F}\varepsilon(u_2), \varepsilon(v))_{\mathcal{H}} + (\mathcal{K}\nabla\theta_1 - \mathcal{K}\nabla\theta_2, \nabla\xi)_H.$$

Then, we conclude

$$\begin{aligned}(Ax_1 - Ax_2, y)_X &\leq M_{\mathcal{F}}\|u_1 - u_2\|_V\|v\|_V + M_{\mathcal{K}}\|\theta_1 - \theta_2\|_Q\|\xi\|_Q \\ &\leq \underbrace{(M_{\mathcal{F}} + M_{\mathcal{K}})}_{=M_A} \|x_1 - x_2\|_X\|y\|_X. \end{aligned} \tag{3.11}$$

From (3.10)–(3.11), we conclude that the operator A is strongly monotone and Lipschitz continuous on X. Moreover, using the definitions (3.2), (3.3) and (3.8), it is easy to verify that the function J^η is continuous. Keeping in mind that U is a nonempty closed convex subset of X and the element $F^\eta \in X$, it comes from standard arguments on variational inequalities that the elliptic inequality (3.9) has unique solution $x^\eta = (u^\eta, \theta^\eta) \in U$. Hence, Problem $(\mathcal{PV}^\eta)$ has unique solution $x^\eta = (u^\eta, \theta^\eta)$, which finishes the second part of Lemma 1. For the last part of Lemma 1, we consider $\eta = (\eta_i)_i$ and $\tilde{\eta} = (\tilde{\eta}_i)_i$ two elements of Y, and let $x^\eta = (u^\eta, \theta^\eta)$ and $x^{\tilde{\eta}} = (u^{\tilde{\eta}}, \theta^{\tilde{\eta}})$ denote their corresponding solution of Problem (PV^η), respectively. Therefore, the inequality (3.9) implies that, for all $y = (v, \xi) \in U$, we have

$$\begin{aligned}(Ax^\eta, y - x^\eta)_X + J^\eta(y) - J^\eta(x^\eta) &\geq (F^\eta, y - x^\eta)_X, \\ (Ax^{\tilde{\eta}}, y - x^{\tilde{\eta}})_X + J^{\tilde{\eta}}(y) - J^{\tilde{\eta}}(x^{\tilde{\eta}}) &\geq (F^{\tilde{\eta}}, y - x^{\tilde{\eta}})_X.\end{aligned}$$

Taking $y = x^{\tilde{\eta}}$ in the first inequality and $y = x^\eta$ in the second inequality, and add the two obtained inequalities to obtain

$$\begin{aligned}(Ax^\eta - Ax^{\tilde{\eta}}, x^\eta - x^{\tilde{\eta}})_X \leq (F^\eta - F^{\tilde{\eta}}, x^\eta - x^{\tilde{\eta}})_X + J^\eta(x^{\tilde{\eta}}) \\ - J^\eta(x^\eta) + J^{\tilde{\eta}}(x^\eta) - J^{\tilde{\eta}}(x^{\tilde{\eta}}).\end{aligned} \tag{3.12}$$

From the definition (3.7) of functional F^η and the assumption $(\mathcal{H}_2)$, we get

$$(F^\eta - F^{\tilde{\eta}}, x^\eta - x^{\tilde{\eta}})_X = (\mathcal{M}\eta_1 - \mathcal{M}\tilde{\eta}_1, \varepsilon(u^\eta) - \varepsilon(u^{\tilde{\eta}}))_{\mathcal{H}}$$
$$\leq \|\mathcal{M}\| \|\eta_1 - \tilde{\eta}_1\|_{L^2(\Omega)} \|u^\eta - u^{\tilde{\eta}}\|_V \leq \frac{\|\mathcal{M}\|}{c_p} \|\eta - \tilde{\eta}\|_Y \|u^\eta - u^{\tilde{\eta}}\|_V. \quad (3.13)$$

Next, using the definitions (3.2), (3.3) and (3.8), we obtain

$$J^\eta(x^{\tilde{\eta}}) - J^\eta(x^\eta) + J^{\tilde{\eta}}(x^\eta) - J^{\tilde{\eta}}(x^{\tilde{\eta}}) = \int_{\Gamma_4} (\eta_2 - \tilde{\eta}_2)(u_\nu^{\tilde{\eta}} - u_\nu^\eta)\, da$$
$$+ \int_{\Gamma_4} (\eta_3 - \tilde{\eta}_3)(\|u_\tau^{\tilde{\eta}}\| - \|u_\tau^\eta\|)\, da + \int_{\Gamma_3} (\eta_4 - \tilde{\eta}_4)(\theta^{\tilde{\eta}} - \theta^\eta) da$$
$$\leq c_1 \|\eta_2 - \tilde{\eta}_2\|_{L^2(\Gamma_4)} \|u^{\tilde{\eta}} - u^\eta\|_V + c_1 \|\eta_3 - \tilde{\eta}_3\|_{L^2(\Gamma_4)} \|u^{\tilde{\eta}} - u^\eta\|_V$$
$$+ c_2 \|\eta_4 - \tilde{\eta}_4\|_{L^2(\Gamma_3)} \|\theta^{\tilde{\eta}} - \theta^\eta\|_Q \leq (2c_1 + c_2) \|\eta - \tilde{\eta}\|_Y \|x^\eta - x^{\tilde{\eta}}\|_X. \quad (3.14)$$

Finally, we combine (3.12)–(3.14) and (3.10) to deduce

$$\|(u^\eta, \theta^\eta) - (u^{\tilde{\eta}}, \theta^{\tilde{\eta}})\|_X \leq c \|\eta - \tilde{\eta}\|_Y,$$

where $c = \big((2c_1 + c_2) + \frac{\|\mathcal{M}\|}{c_p}\big)/m_A > 0$, and hence Lemma 1 is proved. □

In the next step, we consider the operator $\Lambda : Y \to Y$ defined as follows

$$\Lambda(\eta) = (\Lambda_1(\eta), \Lambda_2(\eta), \Lambda_3(\eta), \Lambda_4(\eta)), \quad (3.15)$$
$$\Lambda_1(\eta) = \theta^\eta, \quad \Lambda_2(\eta) = p_\nu(u_\nu^\eta - g)\, h_\nu(\theta^\eta - \theta_F),$$
$$\Lambda_3(\eta) = p_\tau(u_\nu^\eta - g)\, h_\tau(\theta^\eta - \theta_F), \quad \Lambda_4(\eta) = k_T(u_\nu^\eta - g)\, \varphi_L(\theta^\eta - \theta_F),$$

where (u^η, θ^η) is the unique solution of Problem $(\mathcal{PV}^\eta)$ corresponding to η. We will prove that the operator Λ has fixed point and to this end, we consider the following closed convex subsets

$$\mathrm{E}_1 = \{\xi \in Q, \ \ \|\xi\|_Q \leq k_1\}, \quad \mathrm{E}_2 = \{\omega \in L^2(\Gamma_4), \ \ \|\omega\|_{L^2(\Gamma_4)} \leq k_2\},$$
$$\mathrm{E}_3 = \{\omega \in L^2(\Gamma_4), \ \ \|\omega\|_{L^2(\Gamma_4)} \leq k_3\}, \quad \mathrm{E}_4 = \{\eta \in L^2(\Gamma_3), \ \ \|\eta\|_{L^2(\Gamma_3)} \leq k_4\},$$

where the nonnegative constants k_1, k_2, k_3 and k_4 are given by

$$k_1 = (\|q\|_Q + c_2 k_4)/m_{\mathcal{K}}, \quad k_2 = M_{p_\nu} M_{h_\nu} meas(\Gamma_4)^{\frac{1}{2}},$$
$$k_3 = M_{p_\tau} M_{h_\tau} meas(\Gamma_4)^{\frac{1}{2}}, \quad k_4 = M_{k_T} L\, meas(\Gamma_3)^{\frac{1}{2}}.$$

Then, we consider a nonempty, convex and closed subset $\mathrm{E} = \prod_{i=1}^{4} \mathrm{E}_i$ of Y.

Lemma 2. *The operator Λ defined by* (3.15) *has at least one fixed point.*

Proof. For $\eta = (\eta_i)_i \in \mathrm{E}$ given, let (u^η, θ^η) denote the unique solution of Problem $(\mathcal{PV}^\eta)$ corresponding to η. Then, it comes from assumptions $(\mathcal{H}_3)(a)$,

$(\mathcal{H}_4)(a)$ and $(\mathcal{H}_5)(a)$ that

$$\|p_\nu(u_\nu^\eta - g)\, h_\nu(\theta^\eta - \theta_F)\|_{L^2(\Gamma_4)} \le M_{p_\nu} M_{h_\nu} meas(\Gamma_4)^{\frac{1}{2}} = k_2, \tag{3.16}$$
$$\|p_\tau(u_\nu^\eta - g)\, h_\tau(\theta^\eta - \theta_F)\|_{L^2(\Gamma_4)} \le M_{p_\tau} M_{h_\tau} meas(\Gamma_4)^{\frac{1}{2}} = k_3, \tag{3.17}$$
$$\|k_T(u_\nu^\eta - g)\, \varphi_L(\theta^\eta - \theta_F)\|_{L^2(\Gamma_3)} \le M_{k_T} L\, meas(\Gamma_3)^{\frac{1}{2}} = k_4. \tag{3.18}$$

On the other hand, we take $\xi = 0$ in the inequality (3.5) to obtain

$$(\mathcal{K}\nabla\theta^\eta, \nabla\theta^\eta)_H + \ell^\eta(\theta^\eta) \le (q, \theta^\eta)_Q. \tag{3.19}$$

Using the definition (3.3) of the mapping ℓ^η, we have

$$|\ell^\eta(\theta^\eta)| \le c_2 \|\eta_4\|_{L^2(\Gamma_3)} \|\|\theta^\eta\|_Q. \tag{3.20}$$

We combine the hypothesis $(\mathcal{H}_1)$ and the inequalities (3.19) and (3.20) to get

$$m_{\mathcal{K}} \|\theta^\eta\|_Q^2 \le \|q\|_Q \|\theta^\eta\|_Q + c_2 \|\eta_4\|_{L^2(\Gamma_3)} \|\|\theta^\eta\|_Q,$$

which leads to the following inequality

$$\|\theta^\eta\|_Q \le \frac{1}{m_{\mathcal{K}}} (\|q\|_Q + c_2 k_4) = k_1. \tag{3.21}$$

From (3.16)–(3.18) and (3.21), we deduce that Λ is an operator of E into itself. We recall that E is a nonempty convex and closed subset of a reflexive space Y. Then, E is weakly compact. Using the continuity of p_ν, p_τ, h_ν, h_τ, k_T and φ_L, and Lemma 1, we deduce that Λ is weakly continuous. Then, by Schauder's fixed point theorem, the operator Λ has at least one fixed point. □

Now, we have all the ingredients to provide the proof of Theorem 1.
Existence. Let η^* be the fixed point of Λ, we denote by $x^* = (u^*, \theta^*)$, the solution of Problem $(\mathcal{PV}^\eta)$ for $\eta = \eta^*$. The definition (3.15) of the operator Λ implies that $x^* = (u^*, \theta^*)$ satisfies Problem $(\mathcal{PV})$ and that leads to the existence part of Theorem 1.
Uniqueness. Let (u_1, θ_1) and (u_2, θ_2) denote two solutions of Problem $(\mathcal{PV})$. Then, it follows from (2.17) that, for all $v \in V$, we have

$$\begin{aligned}(\mathcal{F}\varepsilon(u_1), \varepsilon(v) - \varepsilon(u_1))_{\mathcal{H}} - (\mathcal{M}\theta_1, \varepsilon(v) - \varepsilon(u_1))_{\mathcal{H}} \\ + j_S(v) - j_S(u_1) + j(u_1, \theta_1, v) - j(u_1, \theta_1, u_1) \ge (f, v - u_1)_V,\end{aligned} \tag{3.22}$$
$$\begin{aligned}(\mathcal{F}\varepsilon(u_2), \varepsilon(v) - \varepsilon(u_2))_{\mathcal{H}} - (\mathcal{M}\theta_2, \varepsilon(v) - \varepsilon(u_2))_{\mathcal{H}} \\ + j_S(v) - j_S(u_2) + j(u_2, \theta_2, v) - j(u_2, \theta_2, u_2) \ge (f, v - u_2)_V.\end{aligned} \tag{3.23}$$

After taking $v = u_2$ in (3.22) and $v = u_1$ in (3.23), we add the two obtained inequalities to get

$$\begin{aligned}(\mathcal{F}\varepsilon(u_1) - \varepsilon(u_2), \varepsilon(u_1) - \varepsilon(u_2))_{\mathcal{H}} \le (\mathcal{M}\theta_1 - \mathcal{M}\theta_2, \varepsilon(u_1) - \varepsilon(u_2))_{\mathcal{H}} \\ + j(u_1, \theta_1, u_2) - j(u_1, \theta_1, u_1) + j(u_2, \theta_2, u_1) - j(u_2, \theta_2, u_2).\end{aligned} \tag{3.24}$$

In addition, it comes from the inequality (2.18) that for all $\xi \in W$, we have

$$(\mathcal{K}\nabla\theta_1, \nabla\xi - \nabla\theta_1)_H + l(u_1, \theta_1, \xi - \theta_1) \geq (q, \xi - \theta_1)_Q, \tag{3.25}$$
$$(\mathcal{K}\nabla\theta_2, \nabla\xi - \nabla\theta_2)_H + l(u_2, \theta_2, \xi - \theta_2) \geq (q, \xi - \theta_2)_Q. \tag{3.26}$$

Taking $\xi = \theta_2$ in (3.25), $\xi = \theta_1$ in (3.26), we add obtained inequalities to find

$$(\mathcal{K}\nabla\theta_1 - \mathcal{K}\nabla\theta_2, \nabla\theta_1 - \nabla\theta_2)_H \leq l(u_1, \theta_1, \theta_2 - \theta_1) - l(u_2, \theta_2, \theta_2 - \theta_1). \tag{3.27}$$

Therefore, we combine the two inequalities (3.24) and (3.27) to conclude

$$(\mathcal{F}\varepsilon(u_1) - \varepsilon(u_2), \varepsilon(u_1) - \varepsilon(u_2))_{\mathcal{H}} + (\mathcal{K}\nabla\theta_1 - \mathcal{K}\nabla\theta_2, \nabla\theta_1 - \nabla\theta_2)_H \leq M, \tag{3.28}$$

where the constant $M = M_1 + M_2 + M_3$ is defined by the following expressions

$$\begin{aligned} M_1 &= (\mathcal{M}\theta_1 - \mathcal{M}\theta_2, \varepsilon(u_1) - \varepsilon(u_2))_{\mathcal{H}}, \\ M_2 &= l(u_1, \theta_1, \theta_2 - \theta_1) - l(u_2, \theta_2, \theta_2 - \theta_1), \\ M_3 &= j(u_1, \theta_1, u_2) - j(u_1, \theta_1, u_1) + j(u_2, \theta_2, u_1) - j(u_2, \theta_2, u_2). \end{aligned}$$

Using the assumption $(\mathcal{H}_2)$ and the Friedrichs-Poincaré inequality (2.11) to obtain

$$\begin{aligned} M_1 &\leq \|\mathcal{M}\theta_1 - \mathcal{M}\theta_2\|_{\mathcal{H}} \|\varepsilon(u_1) - \varepsilon(u_2)\|_{\mathcal{H}} \\ &\leq \frac{1}{c_p}\|\mathcal{M}\|\,\|\theta_1 - \theta_2\|_Q \|u_1 - u_2\|_V \leq \frac{1}{2c_p}\|\mathcal{M}\|\big(\|\theta_1 - \theta_2\|_Q^2 + \|u_1 - u_2\|_V^2\big). \end{aligned}$$

Keeping in mind that φ_L is L-bounded and 1-Lipschitz function, we use the definition (2.14), the assumption $(\mathcal{H}_5)$ and the Sobolev trace inequalities (2.12) and (2.13) to deduce

$$\begin{aligned} M_2 = &\int_{\Gamma_3} (k_T(u_{1\nu} - g) - k_T(u_{2\nu} - g))\,\varphi_L(\theta_1 - \theta_F)(\theta_2 - \theta_1)\,da \\ &\quad + \int_{\Gamma_3} k_T(u_{2\nu} - g)\,(\varphi_L(\theta_1 - \theta_F) - \varphi_L(\theta_2 - \theta_F))(\theta_2 - \theta_1)\,da \\ &\leq c_1\,c_2\,L\,L_{k_T}\,\|u_1 - u_2\|_V \|\theta_1 - \theta_2\|_Q + c_2^2\,M_{k_T}\,\|\theta_1 - \theta_2\|_Q^2 \\ &\leq \frac{c_1 c_2}{2}\,L\,L_{k_T}\,\big(\|u_1 - u_2\|_V^2 + \|\theta_1 - \theta_2\|_Q^2\big) + c_2^2 M_{k_T}\|\theta_1 - \theta_2\|_Q^2. \end{aligned} \tag{3.29}$$

Similarly, we use (2.12)–(2.13), (2.15)–(2.16) and assumptions $(\mathcal{H}_3)$-$(\mathcal{H}_4)$ to get

$$\begin{aligned} M_3 &\leq c_1^2 L_{p_\nu}\,M_{h_\nu}\,\|u_1 - u_2\|_V^2 + c_1 c_2 L_{h_\nu} M_{p_\nu}\|\theta_1 - \theta_2\|_Q \|u_1 - u_2\|_V \\ &\quad + c_1^2 L_{p_\tau}\,M_{h_\tau}\,\|u_1 - u_2\|_V^2 + c_1 c_2 L_{h_\tau} M_{p_\tau}\|\theta_1 - \theta_2\|_Q \|u_1 - u_2\|_V \\ &\leq c_1^2 L_{p_\nu}\,M_{h_\nu}\,\|u_1 - u_2\|_V^2 + \frac{c_1 c_2}{2} L_{h_\nu} M_{p_\nu}\big(\|u_1 - u_2\|_V^2 + \|\theta_1 - \theta_2\|_Q^2\big) \\ &\quad + c_1^2 L_{p_\tau} M_{h_\tau}\|u_1 - u_2\|_V^2 + \frac{c_1 c_2}{2} L_{h_\tau} M_{p_\tau}\big(\|u_1 - u_2\|_V^2 + \|\theta_1 - \theta_2\|_Q^2\big). \end{aligned} \tag{3.30}$$

Now, we combine inequalities (3.28)–(3.30) and assumptions $(\mathcal{H}_1)$ to deduce

$$m_{\mathcal{F}}\|u_1-u_2\|_V^2 + m_{\mathcal{K}}\|\theta_1-\theta_2\|_Q^2 \leq \max\Big(\frac{1}{2c_p},\frac{c_1c_2}{2},c_1^2\Big)(L_1\|u_1-u_2\|_V^2+L_2\|\theta_1-\theta_2\|_Q^2),$$

where the two nonnegative constants L_1 and L_2 are given by

$$L_1 = (\|\mathcal{M}\| + LL_{k_T} + L_{p_\nu}M_{h_\nu} + L_{p_\tau}M_{h_\tau} + L_{h_\nu}M_{p_\nu} + L_{h_\tau}M_{p_\tau}), \tag{3.31}$$

$$L_2 = (\|\mathcal{M}\| + LL_{k_T} + M_{k_T} + L_{h_\nu}M_{p_\nu} + L_{h_\tau}M_{p_\tau}). \tag{3.32}$$

Recalling the smallness conditions (3.1), we conclude

$$\|u_1-u_2\|_V^2 + \|\theta_1-\theta_2\|_Q^2 \leq 0,$$

which implies $u_1 = u_2$ and $\theta_1 = \theta_2$. Thus, the uniqueness part is proved.

4 Convergence results

In this section, we deal with the continuous dependence of the solution of Problem $(\mathcal{PV})$ on the data. To this end, we assume $(\mathcal{H}_1)$–$(\mathcal{H}_7)$ and the smallness conditions (3.1) holds. Then, according to Theorem 1, Problem $(\mathcal{PV})$ has a unique solution (u,θ). Since the solution (u,θ) depends on the data f_0, f_2, q_0, S, θ_F and g, we denote it by $(u,\theta)=\big(u(f_0,f_2,q_0,S,\theta_F,g),\theta(f_0,f_2,q_0,S,\theta_F,g)\big)$. Moreover, we consider in the sequel, a perturbation f_{0n}, f_{2n}, q_{0n}, S_n, θ_{F_n} and g_n of the elements f_0, f_2, q_0, S, θ_F and g, respectively.

For each $n \in \mathbb{N}$, we consider the subset W_n of Q given by

$$W_n = \{\xi \in Q\ ,\quad \xi \leq \theta_{F_n} \quad \text{on} \quad \Gamma_4\},$$

functionals $j_{S_n} : V \to \mathbb{R}$, $j_n : V \times Q \times V \to \mathbb{R}$, $l_n : V \times Q \times Q \to \mathbb{R}$ defined by

$$j_{S_n}(v) = \int_{\Gamma_3} S_n\|v_\tau\|\,da, \tag{4.1}$$

$$l_n(u,\theta,\xi) = \int_{\Gamma_3} k_T(u_\nu - g_n)\varphi_L(\theta-\theta_{F_n})\xi\,da, \tag{4.2}$$

$$j_n(u,\theta,v) = \underbrace{\int_{\Gamma_4} p_\nu(u_\nu-g_n)h_\nu(\theta-\theta_{F_n})v_\nu\,da}_{=j_{n\nu}(u,\theta,v)} \tag{4.3}$$

$$+\underbrace{\int_{\Gamma_4} p_\tau(u_\nu-g_n)h_\tau(\theta-\theta_{F_n})\|v_\tau\|\,da}_{=j_{n\tau}(u,\theta,v)} \tag{4.4}$$

and the elements f_n and q_n defined for all $v \in V$ and $\xi \in Q$ by

$$(f_n,v)_V = \int_\Omega f_{0n}\cdot v\,dx + \int_{\Gamma_2} f_{2n}\cdot v\,da - \int_{\Gamma_3} S_n\cdot v_\nu\,da, \tag{4.5}$$

$$(q_n,\xi)_Q = \int_\Omega q_{0n}\xi\,dx. \tag{4.6}$$

Then, we introduce the following perturbation of Problem $(\mathcal{PV})$

Problem $[\mathcal{PV}_n]$. Find $(u_n, \theta_n) \in V \times W_n$ such that

$$\begin{aligned}(\mathcal{F}\varepsilon(u_n), \varepsilon(v) - \varepsilon(u_n))_{\mathcal{H}} - (\mathcal{M}\theta_n, \varepsilon(v) - \varepsilon(u_n))_{\mathcal{H}} + j_{S_n}(v) \\ - j_{S_n}(u_n) + j_n(u_n, \theta_n, v) - j_n(u_n, \theta_n, u_n) \geq (f_n, v - u_n)_V \quad \forall v \in V,\end{aligned} \tag{4.7}$$

$$(\mathcal{K}\nabla\theta_n, \nabla\xi - \nabla\theta_n)_H + l_n(u_n, \theta_n, \xi - \theta_n) \geq (q_n, \xi - \theta_n)_Q \ \forall \xi \in Q. \tag{4.8}$$

As done to prove Theorem 1, we can get that for each $n \in \mathbb{N}$, Problem $(\mathcal{PV}_n)$ has a unique solution $(u_n, \theta_n) \in V \times W_n$, that we can also write as follows

$$(u_n, \theta_n) = \big(u_n(f_{0n}, f_{2n}, q_{0n}, S, \theta_{F_n}, g_n), \theta_n(f_{0n}, f_{2n}, q_{0n}, S, \theta_{F_n}, g_n)\big).$$

Now, we state the main convergence result of this section.

Theorem 2. *Assume that the following convergences hold*

$$f_{0n} \rightharpoonup f_0 \quad in \quad L^2(\Omega)^d, \tag{4.9}$$

$$f_{2n} \rightharpoonup f_2 \quad in \quad L^2(\Gamma_2)^d, \tag{4.10}$$

$$q_{0n} \rightharpoonup q_0 \quad in \quad L^2(\Omega), \tag{4.11}$$

$$S_n \rightharpoonup S \quad in \quad L^2(\Gamma_3), \tag{4.12}$$

$$g_n \to g \quad in \quad L^2(\Gamma_3 \cup \Gamma_4), \tag{4.13}$$

$$\theta_{F_n} \to \theta_F \quad in \quad \mathbb{R}. \tag{4.14}$$

Then, the solution (u_n, θ_n) *of Problem* $(\mathcal{PV}_n)$ *converges to the solution* (u, θ) *of Problem* $(\mathcal{PV})$*, i.e.,*

$$u_n \to u \quad in \quad V, \qquad \theta_n \to \theta \quad in \quad Q. \tag{4.15}$$

The convergence result in Theorem 2 is important from the mechanical point of view, since it shows that the weak solution of the contact Problem $(\mathcal{P})$ depends continuously on the data. The proof of Theorem 2 will be carried out in several steps. We start by considering the following intermediate problem.

Problem $[\overline{\mathcal{PV}}_n]$. Find $(\overline{u}_n, \overline{\theta}_n) \in V \times W$ such that for all $(v, \xi) \in V \times W$, we have

$$\begin{aligned}(\mathcal{F}\varepsilon(\overline{u}_n), \varepsilon(v) - \varepsilon(\overline{u}_n))_{\mathcal{H}} - (\mathcal{M}\overline{\theta}_n, \varepsilon(v) - \varepsilon(\overline{u}_n))_{\mathcal{H}} \\ + j_{S_n}(v) - j_{S_n}(\overline{u}_n) + j_n(\overline{u}_n, \overline{\theta}_n, v) - j_n(\overline{u}_n, \overline{\theta}_n, \overline{u}_n) \geq (f_n, v - \overline{u}_n)_V,\end{aligned} \tag{4.16}$$

$$(\mathcal{K}\nabla\overline{\theta}_n, \nabla\xi - \nabla\overline{\theta}_n)_H + l_n(\overline{u}_n, \overline{\theta}_n, \xi - \overline{\theta}_n) \geq (q_n, \xi - \overline{\theta}_n)_Q. \tag{4.17}$$

The difference between the two previous problems is that, in Problem $(\overline{\mathcal{PV}}_n)$, we are looking for $\overline{\theta}_n \in W$, while in Problem $(\mathcal{PV}_n)$, we search for $\theta_n \in W_n$. Note that the solvability of Problem $(\overline{\mathcal{PV}}_n)$ is a consequence of Theorem 1. Moreover, we have the following result.

Lemma 3. *Let* (u, θ), (u_n, θ_n) *and* $(\overline{u}_n, \overline{\theta}_n)$ *be the solutions of the problems* $(\mathcal{PV})$, $(\mathcal{PV}_n)$ *and* $(\overline{\mathcal{PV}}_n)$*, respectively. Then, we have*

1. For any $n \in \mathbb{N}$, there exists a constant $\delta > 0$ such that

$$\|u_n\|_V + \|\theta_n\|_Q \leq \delta, \quad \|\overline{u}_n\|_V + \|\overline{\theta}_n\|_Q \leq \delta.$$

2. The sequence $\{(\overline{u}_n, \overline{\theta}_n)\}$ converge weakly to (u, θ) in $V \times Q$, that is

$$\overline{u}_n \rightharpoonup u \quad in \quad V, \quad \overline{\theta}_n \rightharpoonup \theta \quad in \quad Q.$$

3. The sequence $\{(\overline{u}_n, \overline{\theta}_n)\}$ converge strongly to $\{(u, \theta)\}$ in $V \times Q$, that is

$$\overline{u}_n \to u \quad in \quad V, \quad \overline{\theta}_n \to \theta \quad in \quad Q.$$

4. The sequence $\{(\overline{u}_n, \overline{\theta}_n) - (u_n, \theta_n)\}$ converge strongly to zero in $V \times Q$, i.e.,

$$\|\overline{u}_n - u_n\|_V + \|\overline{\theta}_n - \theta_n\|_Q \to 0.$$

Proof. Let $n \in \mathbb{N}$, after taking $v = 0$ in (4.7) and $\xi = 0$ in (4.8), we add the obtained inequalities. Recalling $j_{S_n}(0_V) = j_n(u_n, \theta_n, 0_V) = 0$, we find

$$(\mathcal{F}\varepsilon(u_n),\varepsilon(u_n))_{\mathcal{H}} + (\mathcal{K}\nabla\theta_n, \nabla\theta_n)_H \leq (f_n, u_n)_V + (q_n, \theta_n)_Q \\ - j_{S_n}(u_n) - j_n(u_n, \theta_n, u_n) - l_n(u_n, \theta_n, \theta_n) + (\mathcal{M}\theta_n, \varepsilon(u_n))_{\mathcal{H}}.$$

The definitions of j_{S_n} and $j_{n\tau}$ imply $j_{S_n}(u_n) \geq 0$ and $j_{n\tau}(u_n, \theta_n, u_n) \geq 0$. Then

$$(\mathcal{F}\varepsilon(u_n), \varepsilon(u_n))_{\mathcal{H}} + (\mathcal{K}\nabla\theta_n, \nabla\theta_n)_H \leq (f_n, u_n)_V + (q_n, \theta_n)_Q \\ - j_{n\nu}(u_n, \theta_n, u_n) - l_n(u_n, \theta_n, \theta_n) + (\mathcal{M}\theta_n, \varepsilon(u_n))_{\mathcal{H}}. \tag{4.18}$$

Using the definition (4.5) of f_n and the inequalities (2.10) and (2.12), we find

$$(f_n, u_n)_V \leq \frac{1}{c_k}\|f_{0n}\|_{L^2(\Omega)^d}\|u_n\|_V + c_1\|f_{2n}\|_{L^2(\Gamma_2)^d}\|u_n\|_V + c_1\|S_n\|_{L^2(\Gamma_3)}\|u_n\|_V. \tag{4.19}$$

Next, it follows from (4.9), (4.10) and (4.12) that sequences $\{f_{0n}\} \subset L^2(\Omega)^d$, $\{f_{2n}\} \subset L^2(\Gamma_2)^d$ and $\{S_n\} \subset L^2(\Gamma_3)$ are bounded, i.e., there exist nonnegative constants δ_1, δ_2 and δ_3 such that

$$\|f_{0n}\|_{L^2(\Omega)^d} \leq \delta_1, \quad \|f_{2n}\|_{L^2(\Gamma_2)^d} \leq \delta_2, \quad \|S_n\|_{L^2(\Gamma_3)} \leq \delta_3. \tag{4.20}$$

Then, we combine the two previous inequalities (4.19) and (4.20) to deduce

$$(f_n, u_n)_V \leq (\frac{1}{c_k}\delta_1 + c_1\delta_2 + c_1\delta_3)\,\|u_n\|_V. \tag{4.21}$$

Similarly, the convergence condition (4.11) implies that $\{q_{0n}\} \subset L^2(\Omega)$ is a bounded sequence. Then, there exists a nonnegative constant $\tilde{\delta}_1$ which does not depend on n, such that

$$\|q_{0n}\|_{L^2(\Omega)} \leq \tilde{\delta}_1. \tag{4.22}$$

Using the previous inequality, the definition (4.6) of q_n and (2.11), we find

$$(q_n, \theta_n)_Q \leq \|q_{0n}\|_{L^2(\Omega)} \|\theta_n\|_{L^2(\Omega)} \leq \frac{\tilde{\delta}_1}{c_p} \|\theta_n\|_Q.$$

The definitions (2.9), (4.2) of φ_L and l_n, assumption $(\mathcal{H}_5)$ and (2.13) imply

$$\begin{aligned} |l_n(u_n, \theta_n, \theta_n)| &\leq L M_{k_T} meas(\Gamma_3)^{\frac{1}{2}} \|\theta_n\|_{L^2(\Gamma_3)} \\ &\leq c_2 L M_{k_T} meas(\Gamma_3)^{\frac{1}{2}} \|\theta_n\|_Q. \end{aligned}$$

In addition, the hypotheses $(\mathcal{H}_3)$, $(\mathcal{H}_4)$ and inequality (2.12) lead to

$$\begin{aligned} |j_{n\nu}(u_n, \theta_n, u_n)| &\leq M_{p_\nu} M_{h_\nu} meas(\Gamma_4)^{\frac{1}{2}} \|u_n\|_{L^2(\Gamma_4)^d} \\ &\leq c_1 M_{p_\nu} M_{h_\nu} meas(\Gamma_4)^{\frac{1}{2}} \|u_n\|_V. \end{aligned}$$

Also, it comes from assumption $(\mathcal{H}_2)$ and (2.11) that

$$\begin{aligned} (\mathcal{M}\theta_n, \varepsilon(u_n))_{\mathcal{H}} &\leq \|\mathcal{M}\theta_n\|_{\mathcal{H}} \|\varepsilon(u_n)\|_{\mathcal{H}} \\ &\leq \frac{1}{c_p} \|\mathcal{M}\| \|\theta_n\|_Q \|u_n\|_V \leq \frac{1}{2c_p} \|\mathcal{M}\| \big(\|\theta_n\|_Q^2 + \|u_n\|_V^2\big). \end{aligned} \tag{4.23}$$

Next, we combine (4.18) and (4.21)–(4.23) with the inequality below

$$(\mathcal{F}\varepsilon(u_n), \varepsilon(u_n))_{\mathcal{H}} + (\mathcal{K}\nabla\theta_n, \nabla\theta_n)_H \geq m_{\mathcal{F}} \|u_n\|_V^2 + m_{\mathcal{K}} \|\theta_n\|_Q^2$$

to find that there exist two constants $\tilde{c}_1 > 0$ and $\tilde{c}_2 > 0$ such that

$$\begin{aligned} \big(m_{\mathcal{F}} - \frac{1}{2c_p} \|\mathcal{M}\|\big) \|u_n\|_V^2 + \big(m_{\mathcal{K}} - \frac{1}{2c_p} \|\mathcal{M}\|\big) \|\theta_n\|_Q^2 &\leq (\tilde{c}_1 M_{p_\nu} M_{h_\nu} meas(\Gamma_4)^{\frac{1}{2}} \\ + \|f_n\|_V) \|u_n\|_V + (\tilde{c}_2 L\, M_{k_T} meas(\Gamma_3)^{\frac{1}{2}} + \|q_n\|_Q) \|\theta_n\|_Q. \end{aligned} \tag{4.24}$$

Recalling condition (3.1), we have $m_{\mathcal{F}} - \frac{1}{2c_p}\|\mathcal{M}\| > 0$ and $m_{\mathcal{K}} - \frac{1}{2c_p}\|\mathcal{M}\| > 0$. Then, it comes from (4.24) that there exists a constant $c > 0$ such that

$$\|u_n\|_V^2 + \|\theta_n\|_Q^2 \leq c\big(\|u_n\|_V + \|\theta_n\|_Q\big).$$

Hence, this inequality combined with the fact $(a+b)^2 \leq 2(a^2+b^2)$ for two reals a and b, we conclude that there exists a nonnegative constant δ such that

$$\|u_n\|_V + \|\theta_n\|_Q \leq \delta.$$

In addition, using the same technique, we also deduce

$$\|\overline{u}_n\|_V + \|\overline{\theta}_n\|_Q \leq \delta.$$

Let us now, show that the sequence $\{(\overline{u}_n, \overline{\theta}_n)\}$ converge weakly to (u, θ). It follows from the first part of Lemma 3 that $\{(\overline{u}_n, \overline{\theta}_n)\}$ is bounded sequence in $V \times Q$. Therefore, there exists an element $(\overline{u}, \overline{\theta}) \in V \times Q$ and a subsequence of $\{(\overline{u}_n,\overline{\theta}_n)\}$, denoted again $\{(\overline{u}_n,\overline{\theta}_n)\}$, such that $\{(\overline{u}_n,\overline{\theta}_n)\}$ converge weakly to $(\overline{u},\overline{\theta})$ in $V\times Q$, i.e.,

$$\overline{u}_n \rightharpoonup \overline{u} \quad \text{in} \quad V, \qquad \overline{\theta}_n \rightharpoonup \overline{\theta} \quad \text{in} \quad Q. \tag{4.25}$$

Using compactness result of the embedding of $H^1(\Omega)\hookrightarrow L^2(\Omega)$ [10, Theorem 16.1], we get that $\{(\overline{u}_n,\overline{\theta}_n)\}$ converge strongly to $(\overline{u},\overline{\theta})$ in $L^2(\Omega)^d\times L^2(\Omega)$, i.e.,

$$\overline{u}_n \to \overline{u} \quad \text{in} \quad L^2(\Omega)^d\ , \qquad \overline{\theta}_n \to \overline{\theta} \quad \text{in} \quad L^2(\Omega). \tag{4.26}$$

Since the trace map $\gamma_1: V\to L^2(\Gamma)^d$ and $\gamma_2: Q\to L^2(\Gamma)$ are compacts, then the weak convergence $(\overline{u}_n,\overline{\theta}_n)\rightharpoonup(\overline{u},\overline{\theta})$ in $V\times Q$ leads to the strong convergence $(\overline{u}_n,\overline{\theta}_n)\to(\overline{u},\overline{\theta})$ in $L^2(\Gamma)^d\times L^2(\Gamma)$, i.e.,

$$\overline{u}_n \to \overline{u} \quad \text{in} \quad L^2(\Gamma)^d\ , \qquad \overline{\theta}_n \to \overline{\theta} \quad \text{in} \quad L^2(\Gamma). \tag{4.27}$$

To prove that $(\overline{u},\overline{\theta})=(u,\theta)$, we recall that $V\times W$ is a nonempty closed convex subset of space $V\times Q$ and $\{(\overline{u}_n,\overline{\theta}_n)\}\subset V\times W$. Hence, the convergence (4.25) implies that $(\overline{u},\overline{\theta})\in V\times W$. Then, we take $v=\overline{u}$ in (4.16) and $\xi=\overline{\theta}$ in (4.17), and after adding the two obtained inequalities and using the definition (3.6) of the operator A, we obtain

$$\begin{aligned}(A(\overline{u}_n,\overline{\theta}_n),(\overline{u}_n,\overline{\theta}_n)-(\overline{u},\overline{\theta}))_X\le&(f_n,\overline{u}_n-\overline{u})_V+(q_n,\overline{\theta}_n-\overline{\theta})_Q+j_{S_n}(\overline{u})-j_{S_n}(\overline{u}_n)\\ &+j_n(\overline{u}_n,\overline{\theta}_n,\overline{u})-j_n(\overline{u}_n,\overline{\theta}_n,\overline{u}_n)-l_n(\overline{u}_n,\overline{\theta}_n,\overline{\theta}_n-\overline{\theta})+(\mathcal{M}\overline{\theta}_n,\varepsilon(\overline{u}_n)-\varepsilon(\overline{u}))_{\mathcal{H}}.\end{aligned} \tag{4.28}$$

Moreover, we use the definitions (4.5), (4.6) of f_n and q_n to deduce

$$\begin{aligned}(f_n,\overline{u}_n-\overline{u})_V=&\int_\Omega f_{0n}\cdot\big(\overline{u}_n-\overline{u}\big)dx+\int_{\Gamma_2} f_{2n}\cdot\big(\overline{u}_n-\overline{u}\big)da\\ &-\int_{\Gamma_3} S_n\cdot\big(\overline{u}_{\nu n}-\overline{u}_\nu\big)da\le\|f_{0n}\|_{L^2(\Omega)^d}\,\|\overline{u}_n-\overline{u}\|_{L^2(\Omega)^d}\\ &+\|f_{2n}\|_{L^2(\Gamma_2)^d}\,\|\overline{u}_n-\overline{u}\|_{L^2(\Gamma_2)^d}+\|S_n\|_{L^2(\Gamma_3)}\,\|\overline{u}_n-\overline{u}\|_{L^2(\Gamma_3)^d},\\ (q_n,\overline{\theta}_n-\overline{\theta})_Q=&\int_\Omega q_{0n}\,\big(\overline{\theta}_n-\overline{\theta}\big)\,dx\le\|q_{0n}\|_{L^2(\Omega)}\|\overline{\theta}_n-\overline{\theta}\|_{L^2(\Omega)}.\end{aligned}$$

From the conditions (4.20), (4.22) and the convergences (4.26)–(4.27), we get

$$(f_n,\overline{u}_n-\overline{u})_V\to 0, \quad (q_n,\overline{\theta}_n-\overline{\theta})_Q\to 0. \tag{4.29}$$

From the definitions (4.1)–(4.3) and assumptions $(\mathcal{H}_3)$–$(\mathcal{H}_4)$, we have

$$j_{S_n}(\overline{u})-j_{S_n}(\overline{u}_n)\le\|S_n\|_{L^2(\Gamma_3)}\|\overline{u}-\overline{u}_n\|_{L^2(\Gamma_3)^d}, \tag{4.30}$$

$$l_n(\overline{u}_n,\overline{\theta}_n,\overline{\theta}_n-\overline{\theta})\le M_{k_T}L\,meas(\Gamma_3)^{\frac12}\,\|\overline{\theta}_n-\overline{\theta}\|_{L^2(\Gamma_3)}, \tag{4.31}$$

$$\begin{aligned}j_n(\overline{u}_n,\overline{\theta}_n,\overline{u})-j_n(\overline{u}_n,\overline{\theta}_n,\overline{u}_n)\le&M_{p_\nu}M_{h_\nu}meas(\Gamma_4)^{\frac12}\|\overline{u}-\overline{u}_n\|_{L^2(\Gamma_4)^d}\\ &+M_{p_\tau}M_{h_\tau}meas(\Gamma_4)^{\frac12}\|\overline{u}-\overline{u}_n\|_{L^2(\Gamma_4)^d}.\end{aligned} \tag{4.32}$$

Therefore, the convergence condition (4.27), combined with (4.20) and (4.30)–(4.32), leads to

$$j_{S_n}(\overline{u}) - j_{S_n}(\overline{u}_n) \to 0, \tag{4.33}$$

$$j_n(\overline{u}_n, \overline{\theta}_n, \overline{u}) - j_n(\overline{u}_n, \overline{\theta}_n, \overline{u}_n) \to 0, \tag{4.34}$$

$$l_n(\overline{u}_n, \overline{\theta}_n, \overline{\theta}_n - \overline{\theta}) \to 0. \tag{4.35}$$

The operator ε is linear, then $\varepsilon(\overline{u}_n) \to \varepsilon(\overline{u})$ in $\mathcal{H}$, and by boundedness of the sequence $\{\overline{\theta}_n\}$, we get

$$(\mathcal{M}\overline{\theta}_n, \varepsilon(\overline{u}_n) - \varepsilon(\overline{u}))_{\mathcal{H}} \to 0. \tag{4.36}$$

Then, we use (4.28), convergences (4.29) and (4.33)–(4.36) to obtain

$$\limsup (A(\overline{u}_n, \overline{\theta}_n), (\overline{u}_n, \overline{\theta}_n) - (\overline{u}, \overline{\theta}))_X \le 0. \tag{4.37}$$

The inequality (4.37) combined with (3.11) implies that A is a pseudomonotone operator. Thus, for all $(v, \xi) \in V \times Q$, we have

$$\liminf (A(\overline{u}_n, \overline{\theta}_n), (\overline{u}_n, \overline{\theta}_n) - (v, \xi))_X \ge (A(\overline{u}, \overline{\theta}), (\overline{u}, \overline{\theta}) - (v, \xi))_X.$$

Therefore, we add the two inequalities of Problem $(\overline{\mathcal{PV}}_n)$ and use the definition of the operator A to get

$$\begin{aligned}(A(\overline{u}_n, \overline{\theta}_n), (\overline{u}_n, \overline{\theta}_n) - (v, \xi))_X \le (f_n, \overline{u}_n - v)_V + (q_n, \overline{\theta}_n - \xi)_Q + j_{S_n}(v) - j_{S_n}(\overline{u}_n)\\ + j_n(\overline{u}_n, \overline{\theta}_n, v) - j_n(\overline{u}_n, \overline{\theta}_n, \overline{u}_n) - l_n(\overline{u}_n, \overline{\theta}_n, \overline{\theta}_n - \xi) + (\mathcal{M}\overline{\theta}_n, \varepsilon(\overline{u}_n) - \varepsilon(v))_{\mathcal{H}},\end{aligned} \tag{4.38}$$

for all $(v, \xi) \in V \times W$. The inequality (4.38) can be reformulated as follows

$$\begin{aligned}&(A(\overline{u}_n, \overline{\theta}_n), (\overline{u}_n, \overline{\theta}_n) - (v, \xi))_X\\ &\le (f_n, \overline{u} - v)_V + (f_n, \overline{u}_n - \overline{u})_V + (q_n, \overline{\theta} - \xi)_Q + (q_n, \overline{\theta}_n - \overline{\theta})_Q\\ &\quad + j_{S_n}(v) - j_{S_n}(\overline{u}) - (j_{S_n}(\overline{u}_n) - j_{S_n}(\overline{u})) + j_n(\overline{u}_n, \overline{\theta}_n, v)\\ &\quad - j_n(\overline{u}_n, \overline{\theta}_n, \overline{u}) - (j_n(\overline{u}_n, \overline{\theta}_n, \overline{u}_n) - j_n(\overline{u}_n, \overline{\theta}, \overline{u})) - l_n(\overline{u}_n, \overline{\theta}_n, \overline{\theta} - \xi)\\ &\quad - l_n(\overline{u}_n, \overline{\theta}_n, \overline{\theta}_n - \overline{\theta}) + (\mathcal{M}\overline{\theta}_n, \varepsilon(\overline{u}) - \varepsilon(v))_{\mathcal{H}} + (\mathcal{M}\overline{\theta}_n, \varepsilon(\overline{u}_n) - \varepsilon(\overline{u}))_{\mathcal{H}}.\end{aligned}$$

Then, by passing to the limit, we get that for all $(v, \xi) \in V \times W$, we have

$$\begin{aligned}\liminf (A(\overline{u}_n, \overline{\theta}_n), (\overline{u}_n, \overline{\theta}_n) - (v, \xi))_X \le (f, \overline{u} - v)_V + (q, \overline{\theta} - \xi)_Q + j_S(v)\\ - j_S(\overline{u}) + j(\overline{u}, \overline{\theta}, v) - j(\overline{u}, \overline{\theta}, \overline{u}) - l(\overline{u}, \overline{\theta}, \overline{\theta} - \xi) + (\mathcal{M}\overline{\theta}, \varepsilon(\overline{u}) - \varepsilon(v))_{\mathcal{H}},\end{aligned}$$

$$\begin{aligned}&(A(\overline{u}, \overline{\theta}), (\overline{u}, \overline{\theta}) - (v, \xi))_X \le (f, \overline{u} - v)_V + (q, \overline{\theta} - \xi)_Q\\ &+ j_S(v) - j_S(\overline{u}) + j(\overline{u}, \overline{\theta}, v) - j(\overline{u}, \overline{\theta}, \overline{u}) - l(\overline{u}, \overline{\theta}, \overline{\theta} - \xi) + (\mathcal{M}\overline{\theta}, \varepsilon(\overline{u}) - \varepsilon(v))_{\mathcal{H}}.\end{aligned}$$

Then, for all $(v, \xi) \in V \times W$, we have

$$\begin{aligned}(A(\overline{u}, \overline{\theta}), (v, \xi) - (\overline{u}, \overline{\theta}))_X - (\mathcal{M}\overline{\theta}, \varepsilon(v) - \varepsilon(\overline{u}))_{\mathcal{H}} + j_S(v) - j_S(\overline{u})\\ + j(\overline{u}, \overline{\theta}, v) - j(\overline{u}, \overline{\theta}, \overline{u}) + l(\overline{u}, \overline{\theta}, \xi - \overline{\theta}) \ge (f, v - \overline{u})_V + (q, \xi - \overline{\theta})_Q.\end{aligned}$$

Now, we take successively $\xi = \overline{\theta}$ and $v = \overline{u}$ in the previous inequality to get

$$\begin{aligned}
&(\mathcal{F}\varepsilon(\overline{u}), \varepsilon(v) - \varepsilon(\overline{u}))_{\mathcal{H}} - (\mathcal{M}\overline{\theta}, \varepsilon(v) - \varepsilon(\overline{u}))_{\mathcal{H}} \\
&\qquad + j_S(v) - j_S(\overline{u}) + j(\overline{u}, \overline{\theta}, v) - j(\overline{u}, \overline{\theta}, \overline{u}) \geq (f, v - \overline{u})_V, \quad \forall\, v \in V, \\
&(\mathcal{K}\nabla\overline{\theta}, \nabla\xi - \nabla\overline{\theta})_H + l(\overline{u}, \overline{\theta}, \xi - \overline{\theta}) \geq (q, \xi - \overline{\theta})_Q, \quad \forall\, \xi \in W.
\end{aligned}$$

It means that $(\overline{u}, \overline{\theta})$ is also a solution of Problem $(\mathcal{PV})$, and from the uniqueness of the solution of Problem $(\mathcal{PV})$, we conclude that $(\overline{u}, \overline{\theta}) = (u, \theta)$. Then, the sequence $\{(\overline{u}_n, \overline{\theta}_n)\}$ converge weakly to (u, θ) in $V \times Q$.

Now, we move to prove that $\{(\overline{u}_n, \overline{\theta}_n)\}$ converges strongly to (u, θ) in $V \times Q$. We take $v = u$ in (4.16) and $\xi = \theta$ in (4.17) and add the obtained inequalities to get

$$\begin{aligned}
&(\mathcal{F}\varepsilon(\overline{u}_n), \varepsilon(\overline{u}_n) - \varepsilon(u))_{\mathcal{H}} + (\mathcal{K}\nabla\overline{\theta}_n, \nabla\overline{\theta}_n - \nabla\theta)_H \\
&\leq (f_n, \overline{u}_n - u)_V + (q_n, \overline{\theta}_n - \theta)_Q + j_{S_n}(u) - j_{S_n}(\overline{u}_n) + j_n(\overline{u}_n, \overline{\theta}_n, u) \\
&\quad - j_n(\overline{u}_n, \overline{\theta}_n, \overline{u}_n) - l_n(\overline{u}_n, \overline{\theta}_n, \overline{\theta}_n - \theta) + (\mathcal{M}\overline{\theta}_n, \varepsilon(\overline{u}_n) - \varepsilon(u))_{\mathcal{H}},
\end{aligned}$$

i.e.,

$$\begin{aligned}
&(\mathcal{F}\varepsilon(\overline{u}_n) - \mathcal{F}\varepsilon(\overline{u}_n), \varepsilon(\overline{u}_n) - \varepsilon(u))_{\mathcal{H}} + (\mathcal{K}\nabla\overline{\theta}_n - \mathcal{K}\nabla\overline{\theta}, \nabla\overline{\theta}_n - \nabla\theta)_H \\
&\leq (f_n, \overline{u}_n - u)_V + (q_n, \overline{\theta}_n - \theta)_Q + j_{S_n}(u) - j_{S_n}(\overline{u}_n) + j_n(\overline{u}_n, \overline{\theta}_n, u) \\
&\quad - j_n(\overline{u}_n, \overline{\theta}_n, \overline{u}_n) - l_n(\overline{u}_n, \overline{\theta}_n, \overline{\theta}_n - \theta) + (\mathcal{M}\overline{\theta}_n, \varepsilon(\overline{u}_n) - \varepsilon(u))_{\mathcal{H}} \\
&\quad - (\mathcal{F}\varepsilon(\overline{u}_n), \varepsilon(\overline{u}_n) - \varepsilon(u))_{\mathcal{H}} - (\mathcal{K}\nabla\overline{\theta}, \nabla\overline{\theta}_n - \nabla\theta)_H.
\end{aligned} \tag{4.39}$$

Recalling that the sequence $(\overline{u}_n, \overline{\theta}_n)$ converges weakly to (u, θ) in $V \times Q$, then by the same techniques used to find (4.29) and (4.33)–(4.36), we deduce

$$(\mathcal{F}\varepsilon(\overline{u}_n), \varepsilon(\overline{u}_n) - \varepsilon(u))_{\mathcal{H}} \to 0, \quad (\mathcal{K}\nabla\overline{\theta}, \nabla\overline{\theta}_n - \nabla\theta)_H \to 0, \tag{4.40}$$

$$(f_n, \overline{u}_n - u)_V \to 0, \quad (q_n, \overline{\theta}_n - \theta)_Q \to 0, \tag{4.41}$$

$$j_{S_n}(u) - j_{S_n}(\overline{u}_n) \to 0, \quad j_n(\overline{u}_n, \overline{\theta}_n, u) - j_n(\overline{u}_n, \overline{\theta}_n, \overline{u}_n) \to 0, \tag{4.42}$$

$$l_n(\overline{u}_n, \overline{\theta}_n, \overline{\theta}_n - \theta) \to 0, \quad (\mathcal{M}\overline{\theta}_n, \varepsilon(\overline{u}_n) - \varepsilon(u))_{\mathcal{H}} \to 0. \tag{4.43}$$

Next, we combine (4.39), (4.40)–(4.43) and assumption $(\mathcal{H}_1)$ to find

$$\lim_{n\to\infty} \|\overline{u}_n - u\|_V^2 + \|\overline{\theta}_n - \theta\|_Q^2\big) \leq 0.$$

Hence, $\big\{(\overline{u}_n, \overline{\theta}_n)\big\}$ converges strongly to (u, θ) in $V \times Q$.

Let us now prove $\big(\|\overline{u}_n - u_n\|_V + \|\overline{\theta}_n - \theta_n\|_Q\big) \to 0$. We consider a nonnegative real $\alpha_n = \theta_F/\theta_{F_n}$. Using the definitions of W and W_n, it is easy to deduce

$$\alpha_n\theta_n \in W, \quad \overline{\theta}_n/\alpha_n \in W_n,$$

where (u_n, θ_n), $(\overline{u}_n, \overline{\theta}_n)$ are the solution of Problem $(\mathcal{PV}_n)$ and Problem $(\overline{\mathcal{PV}}_n)$, respectively. We take $v = \frac{1}{\alpha_n}\overline{u}_n$ in (4.7) and $v = \alpha_n u_n$ in (4.16), then we get

$$(\mathcal{F}\varepsilon(u_n), \varepsilon(\frac{1}{\alpha_n}\overline{u}_n) - \varepsilon(u_n))_{\mathcal{H}} - (\mathcal{M}\theta_n, \varepsilon(\frac{1}{\alpha_n}\overline{u}_n) - \varepsilon(u_n))_{\mathcal{H}} + j_{S_n}(\frac{1}{\alpha_n}\overline{u}_n)$$
$$- j_{S_n}(u_n) + j_n(u_n, \theta_n, \frac{1}{\alpha_n}\overline{u}_n) - j_n(u_n, \theta_n, u_n) \geq (f_n, \frac{1}{\alpha_n}\overline{u}_n - u_n)_V,$$
$$(\mathcal{F}\varepsilon(\overline{u}_n), \varepsilon(\alpha_n u_n) - \varepsilon(\overline{u}_n))_{\mathcal{H}} - (\mathcal{M}\overline{\theta}_n, \varepsilon(\alpha_n u_n) - \varepsilon(\overline{u}_n))_{\mathcal{H}} + j_{S_n}(\alpha_n u_n)$$
$$- j_{S_n}(\overline{u}_n) + j_n(\overline{u}_n, \overline{\theta}_n, \alpha_n u_n) - j_n(\overline{u}_n, \overline{\theta}_n, \overline{u}_n) \geq (f_n, \alpha_n u_n - \overline{u}_n)_V.$$

Next, we add the two previous inequalities to obtain

$$\begin{aligned}
&(\mathcal{F}\varepsilon(u_n), \varepsilon(u_n) - \varepsilon(\frac{1}{\alpha_n}\overline{u}_n))_{\mathcal{H}} + (\mathcal{F}\varepsilon(\overline{u}_n), \varepsilon(\overline{u}_n) - \varepsilon(\alpha_n u_n))_{\mathcal{H}} \\
&\leq (1-\alpha_n)(f_n, u_n)_V + (1 - \frac{1}{\alpha_n})(f_n, \overline{u}_n)_V + (\alpha_n - 1) j_{S_n}(u_n) \\
&\quad + (\frac{1}{\alpha_n} - 1) j_{S_n}(\overline{u}_n) + (\alpha_n - 1)\, j_n(\overline{u}_n, \overline{\theta}_n, u_n) + (\frac{1}{\alpha_n} - 1)\, j_n(u_n, \theta_n, \overline{u}_n) \\
&\quad + j_n(\overline{u}_n, \overline{\theta}_n, u_n) - j_n(\overline{u}_n, \overline{\theta}_n, \overline{u}_n) + j_n(u_n, \theta_n, \overline{u}_n) - j_n(u_n, \theta_n, u_n) \\
&\quad + (1-\alpha_n)(\mathcal{M}\overline{\theta}_n, \varepsilon(u_n))_{\mathcal{H}} + (1 - \frac{1}{\alpha_n})(\mathcal{M}\theta_n, \varepsilon(\overline{u}_n))_{\mathcal{H}} \\
&\quad + (\mathcal{M}\overline{\theta}_n - \mathcal{M}\theta_n, \varepsilon(\overline{u}_n) - \varepsilon(u_n))_{\mathcal{H}}.
\end{aligned} \tag{4.44}$$

Using the assumption $(\mathcal{H}_1)$, it comes from the previous inequality that

$$\begin{aligned}
&(\mathcal{F}\varepsilon(u_n), \varepsilon(u_n) - \varepsilon(\frac{1}{\alpha_n}\overline{u}_n))_{\mathcal{H}} + (\mathcal{F}\varepsilon(\overline{u}_n), \varepsilon(\overline{u}_n) - \varepsilon(\alpha_n\, u_n))_{\mathcal{H}} \\
&= (\mathcal{F}\varepsilon(u_n) - \mathcal{F}\varepsilon(\overline{u}_n), \varepsilon(u_n) - \varepsilon(\overline{u}_n))_{\mathcal{H}} + (1-\alpha_n)(\mathcal{F}\varepsilon(\overline{u}_n), \varepsilon(u_n))_{\mathcal{H}} \\
&\quad + (1 - \frac{1}{\alpha_n})(\mathcal{F}\varepsilon(u_n), \varepsilon(\overline{u}_n))_{\mathcal{H}} \geq m_{\mathcal{F}} \|u_n - \overline{u}_n\|_V^2 \\
&\quad - |1-\alpha_n|\, M_{\mathcal{F}}\, \|\overline{u}_n\|_V \|u_n\|_V - |1 - \frac{1}{\alpha_n}|\, M_{\mathcal{F}}\, \|\overline{u}_n\|_V \|u_n\|_V.
\end{aligned} \tag{4.45}$$

Remembering that f_n, S_n, $\overline{u}_n$, u_n, $\overline{\theta}_n$ and θ_n are all bounded, we use hypotheses $(\mathcal{H}_1)$–$(\mathcal{H}_5)$ and inequalities (4.44) and (4.45) to get that there exists a nonnegative constant $\tilde{c}_1$ such that

$$\begin{aligned}
&m_{\mathcal{F}} \|u_n - \overline{u}_n\|_V^2 \\
&\leq \tilde{c}_1\, (|1-\alpha_n| + |(1 - \frac{1}{\alpha_n}|) + j_n(\overline{u}_n, \overline{\theta}_n, u_n) - j_n(\overline{u}_n, \overline{\theta}_n, \overline{u}_n) \\
&\quad + j_n(u_n, \theta_n, \overline{u}_n) - j_n(u_n, \theta_n, u_n) + (\mathcal{M}\overline{\theta}_n - \mathcal{M}\theta_n, \varepsilon(\overline{u}_n) - \varepsilon(u_n))_{\mathcal{H}}.
\end{aligned} \tag{4.46}$$

Moreover, by the same arguments used to prove (3.30), we can deduce that

$$\begin{aligned}
&|j_n(\overline{u}_n, \overline{\theta}_n, u_n) - j_n(\overline{u}_n, \overline{\theta}_n, \overline{u}_n) + j_n(u_n, \theta_n, \overline{u}_n) - j_n(u_n, \theta_n, u_n)| \\
&\leq c_1^2 L_{p_\nu} M_{h_\nu} \|\overline{u}_n - u_n\|_V^2 + \frac{c_1 c_2}{2} L_{h_\nu} M_{p_\nu} \big(\|\overline{u}_n - u_n\|_V^2 + \|\overline{\theta}_n - \theta_n\|_Q^2\big) \\
&+ c_1^2 L_{p_\tau} M_{h_\tau} \|\overline{u}_n - u_n\|_V^2 + \frac{c_1 c_2}{2} L_{h_\tau} M_{p_\tau} \big(\|\overline{u}_n - u_n\|_V^2 + \|\overline{\theta}_n - \theta_n\|_Q^2\big).
\end{aligned} \tag{4.47}$$

Hence, by combining (4.46)–(4.47) and assumption $(\mathcal{H}_2)$, we get

$$\begin{aligned} & m_{\mathcal{F}}\,\|u_n-\overline{u}_n\|_V^2 \\ & \leq \tilde{c}_1\left(|1-\alpha_n|+|1-\frac{1}{\alpha_n}|\right)+c_1^2L_{p_\nu}M_{h_\nu}\|\overline{u}_n-u_n\|_V^2 \\ & \quad+\frac{c_1c_2}{2}L_{h_\nu}M_{p_\nu}\left(\|\overline{u}_n-u_n\|_V^2+\|\overline{\theta}_n-\theta_n\|_Q^2\right)+c_1^2L_{p_\tau}M_{h_\tau}\|\overline{u}_n-u_n\|_V^2 \\ & \quad+\frac{c_1c_2}{2}L_{h_\tau}M_{p_\tau}\left(\|\overline{u}_n-u_n\|_V^2+\|\overline{\theta}_n-\theta_n\|_Q^2\right) \\ & \quad+\frac{1}{2c_p}\|\mathcal{M}\|\left(\|\overline{\theta}_n-\theta_n\|_Q^2+\|\overline{u}_n-u_n\|_V^2\right). \end{aligned} \tag{4.48}$$

Now, we take $\xi=\alpha_n\theta_n\in W$ and $\xi=\frac{1}{\alpha_n}\overline{\theta}_n\in W_n$ in (4.17) and (4.8), respectively, to obtain

$$\begin{aligned} (\mathcal{K}\nabla\theta_n,\frac{1}{\alpha_n}\nabla\overline{\theta}_n-\nabla\theta_n)_H+l_n(u_n,\theta_n,\frac{1}{\alpha_n}\overline{\theta}_n-\theta_n)\geq(q_n,\frac{1}{\alpha_n}\overline{\theta}_n-\theta_n)_Q, \\ (\mathcal{K}\nabla\overline{\theta}_n,\alpha_n\nabla\theta_n-\nabla\overline{\theta}_n)_H+l_n(\overline{u}_n,\overline{\theta}_n,\alpha_n\theta_n-\overline{\theta}_n)\geq(q_n,\alpha_n\theta_n-\overline{\theta}_n)_Q. \end{aligned}$$

Then, we add the two previous inequalities to deduce

$$\begin{aligned} & (\mathcal{K}\nabla\theta_n,\nabla\theta_n-\frac{1}{\alpha_n}\nabla\overline{\theta}_n)_H+(\mathcal{K}\nabla\overline{\theta}_n,\nabla\overline{\theta}_n-\alpha_n\nabla\theta_n)_H \\ & \leq(1-\alpha_n)(q_n,\theta_n)_Q+(1-\frac{1}{\alpha_n})(q_n,\overline{\theta}_n)_Q \\ & \quad+(\alpha_n-1)\,l_n(\overline{u}_n,\overline{\theta}_n,\theta_n)+(\frac{1}{\alpha_n}-1)\,l_n(u_n,\theta_n,\overline{\theta}_n) \\ & \quad+l_n(u_n,\theta_n,\overline{\theta}_n-\theta_n)-l_n(\overline{u}_n,\overline{\theta}_n,\overline{\theta}_n-\theta_n). \end{aligned} \tag{4.49}$$

In addition, it comes from the hypothesis $(\mathcal{H}_1)$ that

$$\begin{aligned} & (\mathcal{K}\nabla\theta_n,\nabla\theta_n-\frac{1}{\alpha_n}\nabla\overline{\theta}_n)_H+(\mathcal{K}\nabla\overline{\theta}_n,\nabla\overline{\theta}_n-\alpha_n\nabla\theta_n)_H \\ & =(\mathcal{K}\nabla\theta_n-\mathcal{K}\nabla\overline{\theta}_n,\nabla\theta_n-\nabla\overline{\theta}_n)_H+(1-\alpha_n)\,(\mathcal{K}\nabla\overline{\theta}_n,\nabla\theta_n)_H \\ & \quad+(1-\frac{1}{\alpha_n})\,(\mathcal{K}\nabla\theta_n,\nabla\overline{\theta}_n)_H \\ & \geq m_{\mathcal{K}}\|\theta_n-\overline{\theta}_n\|_Q^2-\left(|1-\alpha_n|+|1-\frac{1}{\alpha_n}|\right)M_{\mathcal{K}}\|\overline{\theta}_n\|_Q\,\|\theta_n\|_Q. \end{aligned}$$

By the same arguments as used to find (3.29), we can get

$$\begin{aligned} & |l_n(u_n,\theta_n,\overline{\theta}_n-\theta_n)-l_n(\overline{u}_n,\overline{\theta}_n,\overline{\theta}_n-\theta_n)| \\ & \leq\frac{c_1c_2}{2}LL_{k_T}\left(\|u_n-\overline{u}_n\|_V^2+\|\overline{\theta}_n-\theta_n\|_Q^2\right)+c_2^2M_{k_T}\|\overline{\theta}_n-\theta_n\|_Q^2. \end{aligned} \tag{4.50}$$

Recalling that q_n, $\overline{\theta}_n$, θ_n, $\overline{u}_n$ and u_n are all bounded, then it follows from the hypotheses $(\mathcal{H}_1)$ and $(\mathcal{H}_5)$, the definition (2.9) of φ_L and the inequalities

(4.49)–(4.50) that, there exists a nonnegative constant $\tilde{c}_2$ such that

$$m_{\mathcal{K}}\|\theta_n - \overline{\theta}_n\|_Q^2 \leq \tilde{c}_2\left(|1-\alpha_n| + |1-\frac{1}{\alpha_n}|\right) + c_2^2 M_{k_T}\|\overline{\theta}_n - \theta_n\|_Q^2 + \frac{c_1c_2}{2}\, L\, L_{k_T}\left(\|u_n - \overline{u}_n\|_V^2 + \|\overline{\theta}_n - \theta_n\|_Q^2\right). \tag{4.51}$$

Next, we combine (4.48) and (4.51) to conclude that there exists a constant $\tilde{c}_3 > 0$ such that

$$m_{\mathcal{F}}\,\|u_n - \overline{u}_n\|_V^2 + m_{\mathcal{K}}\|\overline{\theta}_n - \theta_n\|_Q^2 \leq \tilde{c}_3\left(|1-\alpha_n| + |1-\frac{1}{\alpha_n}|\right) + \max(\frac{1}{2c_p}, \frac{c_1c_2}{2}, c_1^2)\, L_1\,\|u_n - \overline{u}_n\|_V^2 + \max(\frac{1}{2c_p}, c_2^2, \frac{c_1c_2}{2})\, L_2\,\|\overline{\theta}_n - \theta_n\|_Q^2,$$

where the constants L_1 and L_2 are previously defined (see page 12). Keeping in mind conditions (3.1), then there exists a nonnegative constant c such that

$$\|u_n - \overline{u}_n\|_V^2 + \|\overline{\theta}_n - \theta_n\|_Q^2 \leq c\left(|1-\alpha_n| + |1-\frac{1}{\alpha_n}|\right). \tag{4.52}$$

Finally, from (4.14), we get $\alpha_n = \dfrac{\theta_F}{\theta_{F_n}} \to 1$, and then (4.52) leads to

$$\|u_n - \overline{u}_n\|_V + \|\overline{\theta}_n - \theta_n\|_Q \to 0,$$

which concludes the proof of Lemma 3. □

Now, we have all the ingredients to provide the proof of Theorem 2. Let $n \in \mathbb{N}$, we denote by (u,θ), (u_n,θ_n) and $(\overline{u}_n,\overline{\theta}_n)$, the solutions of the problems $(\mathcal{PV})$, $(\mathcal{PV}_n)$ and $(\overline{\mathcal{PV}}_n)$, respectively. We know that

$$\|u_n - u\|_V + \|\theta_n - \theta\|_Q \leq \|u_n - \overline{u}_n\|_V + \|\overline{u}_n - u\|_V + \|\theta_n - \overline{\theta}_n\|_Q + \|\overline{\theta}_n - \theta\|_Q.$$

Hence, it follows from Lemma 3 that (4.15) holds, and thus ends the proof of Theorem 2.

5 Optimization problem

In the previous section, we have seen that for given loading functions f_0, f_2, q_0, g, S and θ_F, Problem $(\mathcal{PV})$ has a unique solution (u,θ). So, each of these quantities could play the role of controlling the inequalities of this Problem.

Now, we would like to study an optimization problem which is described by the following construction. Let β and δ be one or a part of the problem's data such that

$$\beta \cap \delta = \emptyset, \quad \beta \cup \delta = \{f_0, f_2, q_0, g, S, \theta_F\}.$$

To guarantee the conditions of Theorem 2, we assume that $\beta \in T$ and $\delta \in T'$, where T and T' are subsets of two appropriate Hilbert spaces Z and Z'. For a given δ, we want to act through a good choice of β, and then the solution of Problem $(\mathcal{PV})$, which of course depends on the data $\beta \cup \delta$, is now considered as function of β. Hence, we denote it in what follows by $(u(\beta), \theta(\beta))$. Next, we consider a the cost functional $\mathcal{L} : X \to \mathbb{R}$, and the following minimization problem.

Problem $[\mathcal{PO}]$. Given $\delta \in T'$, find $\beta \in T$ such that

$$\mathcal{L}(u(\beta^*), \theta(\beta^*)) = \min_{\beta \in T} \mathcal{L}(u(\beta), \theta(\beta)). \tag{5.1}$$

We note here that for a given $\delta \in T'$, the mapping $\beta \mapsto (\beta, \delta)$ is linear continuous for the strong topologies, and then it is also continuous for the weak topologies.

The main result of this section is stated as follows.

Theorem 3. *We assume that the following hypotheses hold,*

$$T \text{ is a bounded weakly closed subset of the space } Z, \tag{5.2}$$

$$\mathcal{L} : X \to \mathbb{R} \text{ is a lower semicontinuous function.} \tag{5.3}$$

Then, for each $\delta \in T'$, Problem $(\mathcal{PO})$ has at least one solution $\beta^* \in T$.

Proof. For $\delta \in T'$ given, we consider $\vartheta = \inf_{\beta \in T} \mathcal{L}(u(\beta), \theta(\beta))$ and $(\beta_n) \subset T$ the minimizing sequence for the functional $\mathcal{L}$. Then, it comes from the definition of $\mathcal{L}$ that

$$\lim \mathcal{L}(u(\beta_n), \theta(\beta_n)) = \vartheta. \tag{5.4}$$

From hypothesis (5.2), T is bounded subset in Z, and hence (β_n) is a bounded sequence in Z. Thus, there exist $\beta^* \in Z$ and a subsequence of (β_n), still denoted (β_n), such that

$$\beta_n \rightharpoonup \beta^* \quad \text{in} \quad Z. \tag{5.5}$$

Moreover, since $T \subset Z$ is weakly closed, the convergence (5.5) implies

$$\beta^* \in T. \tag{5.6}$$

Then, using the regularity (5.6), the convergence (5.5) and Theorem 2, we obtain

$$(u(\beta_n), \theta(\beta_n)) \to (u(\beta^*), \theta(\beta^*)) \quad \text{in} \quad X.$$

Keeping in mind hypothesis (5.3), we deduce

$$\liminf \mathcal{L}(u(\beta_n), \theta(\beta_n)) \geq \mathcal{L}(x(\beta^*, \delta)). \tag{5.7}$$

Next, we combine the previous inequality (5.7) and (5.4) to get

$$\vartheta \geq \mathcal{L}(u(\beta^*), \theta(\beta^*)). \tag{5.8}$$

In addition, it follows from (5.6) that

$$\vartheta = \inf_{\beta \in T} \mathcal{L}(u(\beta), \theta(\beta)) \leq \mathcal{L}(u(\beta^*), \theta(\beta^*)). \tag{5.9}$$

Finally, we use (5.8) and (5.9) to see that (5.1) holds, and thus concludes the proof. □

We could as well consider various examples of cost function in which we can obtain analogous results without any additional difficulties. For instance, we take two examples of optimization problems for which the existence results provided by Theorem 3.

Example 1. A first example of Problem $(\mathcal{PO})$ can be obtained by taking

$$\beta = (f_2, S, g, \theta_F), \qquad \delta = (f_0, q_0),$$
$$Z = L^2(\Gamma_2)^d \times L^2(\Gamma_3) \times L^2(\Gamma_3 \cup \Gamma_4) \times \mathbb{R}, \qquad Z' = L^2(\Omega)^d \times L^2(\Omega),$$
$$T = \{\beta \in Z\,, \ \ \|\beta\|_Z \le C\}, \qquad T' = Z',$$

where C is a nonnegative constant, and the following cost function

$$\mathcal{L}(v,\xi) = \int_\Omega (\|\sigma(v,\xi)\|^2 + \|q_T(\xi)\|^2)\, dx \quad \forall\, (v,\xi) \in V \times Q,$$

where $\sigma(v,\xi) = \mathcal{F}\varepsilon(v) - \mathcal{M}\xi$ and $q_T(\xi) = -\mathcal{K}\xi$. The mechanical interpretation is the following; given a contact process of the form (2.1)–(2.6), with the data $(f_0, q_0) \in T'$, we are looking for a traction f_2^*, a friction bound S^*, a gap function g^* and a foundation's temperature θ_F^* such that the corresponding stress in the body and heat flux are as small as possible.

We note that T is a bounded weakly closed subset of Z and hence it satisfies condition (5.2). Moreover, since the function $\mathcal{L} : X \to \mathbb{R}$ is continuous, it is a fortiori lower semicontinuous function, and then, it satisfies condition (5.3). Therefore, Theorem 3 guarantees the existence of solutions to the corresponding optimization problem.

Example 2. In second example of Problem $(\mathcal{PO})$, we consider

$$\beta = f_2, \quad \delta = (f_0, q_0, g, \theta_F),$$
$$Z = L^2(\Gamma_2)^d, \quad Z' = L^2(\Omega)^d \times L^2(\Omega) \times L^2(\Gamma_3 \cup \Gamma_4) \times \mathbb{R},$$
$$T = \{\beta \in Z,\ \|\beta\|_Z \le C\},\ T' = \{\delta \in Z',\ g_0 \le g \le g_1 \ \ et \ \ \theta_0 \le \theta_F \le \theta_1\},$$
$$\mathcal{L}(v,\xi) = \int_{\Gamma_4} (\|v_\nu - u_d\|^2 + \|\xi - \theta_d\|^2)\, da, \quad \forall\, (v,\xi) \in V \times Q,$$

where C, g_0, g_1, θ_0 and θ_1 are nonnegative constants such that $g_0 \le g_1$, $\theta_0 \le \theta_1$, $u_d \in L^2(\Gamma_4)$ and $\theta_d \in L^2(\Gamma_4)$ are given. We want to find the surface traction f_2 acting on Γ_2 which leads to the desired displacement field u_d and desired temperature θ_d on the part Γ_4.

It easy to see that T is a bounded weakly closed subset of Z, and hence it satisfies the condition (5.2). In addition, since $\mathcal{L} : X \to \mathbb{R}$ is continuous, it is a fortiori lower semicontinuous, and then it satisfies condition (5.3). Finally, Theorem 3 guarantees the existence of solutions to the corresponding optimization problem.

References

[1] A. Amassad, D. Chenais and C. Fabre. Optimal control of an elastic contact problem involving tresca friction law. *Nonlinear Anal. Theory Methods Appl.*, **48**(8):1107–1135, 2002. https://doi.org/10.1016/S0362-546X(00)00241-8.

[2] A. Amassad, K.L. Kuttler, M. Rochdi and M. Shillor. Quasi-static thermoviscoelastic contact problem with slip dependent friction coefficient. *Math. Comput. Model.*,**36**(7–8):839–854,2002. https://doi.org/10.1016/S0895-7177(02)00231-5.

[3] L. E. Andersson, A. Klarbring, J. R. Barber and M. Ciavarella. On the existence and uniqueness of steady state solutions in thermoelastic contact with frictional heating. In *Proceedings of the Royal Society A Mathematical, Physical and Engineering Sciences*, volume 461, pp. 1261–1282, 2005. https://doi.org/10.1098/rspa.2004.1398.

[4] M. Campo and J. Fernández. Numerical analysis of a quasistatic thermoviscoelastic frictional contact problem. *J. Comput. Mech.*, **35**:459–469, 2005. https://doi.org/10.1007/s00466-004-0635-4.

[5] M. Couderc and M. Sofonea. An elastic frictional contact problem with unilateral constraint. *Mediterr. J. Math.*, **15**(195), 2018. https://doi.org/10.1007/s00009-018-1243-4.

[6] G. Duvaut. Free boundary problem connected with thermoelasticity and unilateral contact. In *Free boundary problems*, volume 2 of *Free boundary problems*, pp. 217–236, Roma, 1980.

[7] C. Eck and J. Jarušek. *Existence of solutions to a nonlinear coupled thermoviscoelastic contact problem with small coulomb friction.* Applied Nonlinear Analysis, Springer, Boston, 2002.

[8] A.S. El-Karamany and M.A. Ezzat. On the boundary integral formulation of thermo-viscoelasticity theory. *International Journal of Engineering Science*, **40**(17):1943–1956, 2002. https://doi.org/10.1016/S0020-7225(02)00043-5.

[9] J.L. Lions. *Contrôle optimal de systèmes gouvernés par des équations aux dérivées partielles.* Dunod, Paris, 1968.

[10] J.L. Lions and E. Magenes. *Problèmes aux limites non homogènes et applications.* Dunod, Paris, 1968.

[11] A. Matei and S. Micu. Boundary optimal control for nonlinear antiplane problems. *Nonlinear Anal. Theory Methods Appl.*, **74**(5):1641–1652, 2011. https://doi.org/10.1016/j.na.2010.10.034.

[12] A. Matei and S. Micu. Boundary optimal control for a frictional contact problem with normal compliance. *Appl. Math. Optim.*, **78**:379–401, 2018. https://doi.org/10.1007/s00245-017-9410-8.

[13] A. Matei, S. Micu and C. Nită. Optimal control for antiplane frictional contact problems involving nonlinearly elastic materials of hencky type. *Math. Mech. Solids*, **23**(3):308–328, 2017. https://doi.org/10.1177/1081286517718605.

[14] F. Mignot. Contrôle dans les inéquations variationnelles elliptiques. *Jou. Func. Anal.*, **22**(2):130–185, 1976. https://doi.org/10.1016/0022-1236(76)90017-3.

[15] F. Mignot and J.P. Puel. Optimal control in some variational inequalities. *SIAM J. Control Optim.*, **22**(3):466–476, 1984. https://doi.org/10.1137/0322028.

[16] P. Neittaanmaki, J. Sprekels and D. Tiba. *Optimization of Elliptic Systems: Theory and Applications.* Springer Monographs in Mathematics. Springer-Verlag, New York, 2006.

[17] M. Sofonea. Optimal control of variational-hemivariational inequalities in reflexive banach spaces. *Appl. Math. Optim.*, **79**:621–646, 2019. https://doi.org/10.1007/s00245-017-9450-0.

[18] M. Sofonea, Y. Xiao and M. Couderc. Optimization problems for a viscoelastic frictional contact problem with unilateral constraints. *Nonlinear Analysis: Real World Applications*, **50**:86–103, 2019. https://doi.org/10.1016/j.nonrwa.2019.04.005.

[19] M. Sofonea and Y.B. Xiao. Boundary optimal control of a nonsmooth frictionless contact problem. *Comput. Math. Appl.*, **38**(1):152–165, 2019. https://doi.org/10.1016/j.camwa.2019.02.027.

[20] M. Sofonea and Y.B. Xiao. Optimization problems for elastic contact models with unilateral constraints. *Z. Angew. Math. Phys.*, **70**(1), 2019. https://doi.org/10.1007/s00033-018-1046-2.

[21] Y.B. Xiao and M. Sofonea. On the optimal control of variational-hemivariational inequalities. *J. Math. Anal. Appl.*, **475**(1):364–384, 2019. https://doi.org/10.1016/j.jmaa.2019.02.046.

[22] J.P. Yvon. *Contrôle optimal de systèmes gouvernés par des inéquations variationnelles.* Rapport Laboria, INRIA, Rocquencourt, France, 1974.