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On Time Periodic Solutions to the Generalized BBM-Burgers Equation with Time-Dependent Periodic External Force

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Abstract. In this paper, we consider the generalized BBM-Burgers equation with periodic external force in $\mathbb{R}^n$. Existence and uniqueness of time periodic solutions that have the same period as the external force are established in some suitable function space for the space dimension $n \geq 3$. Moreover, we also discuss the time asymptotic stability of the time periodic solution. The proof is mainly based on the contraction mapping theorem and continuous argument.

Keywords: generalized BBM-Burgers equation, existence of time periodic solutions, asymptotic stability.

AMS Subject Classification: 35K25; 35B40.

1 Introduction

In this paper, we investigate existence and asymptotic stability of time periodic solutions to the generalized BBM-Burgers equation with time-dependent periodic external force

$$v_t - \alpha \Delta v_t - \beta \Delta v + \gamma \Delta^2 v + \sum_{j=1}^{n} f_j(v)_{x_j} = \Delta g(v) + \sum_{j=1}^{n} h_j(x,t)_{x_j}. \tag{1.1}$$

Here $v = v(x,t)$ is the unknown function of $x \in \mathbb{R}^n$ and $t > 0$, $\alpha > 0$, β and $\gamma > 0$ are constants. For any $j = 1, \ldots, n$, the nonlinear term $f_j(v)$ and $g(v)$ are given smooth function of $v \in \mathbb{R}$ satisfying $g'(0) + \beta > 0$ and the function $h_j(x,t)$ is a periodic function with period T, i.e. $h_j(x, t+T) = h_j(x,t)$.

When $h_j = 0$, Geng and Chen [3] proved existence and the uniqueness of the global generalized solution and the global classical solution for the Cauchy problem of equation (1.1) for $n \leq 3$. The proof is based by existence of local solutions and energy method. Moreover, the decay property of the solution was discussed. When $g = h_j = 0$, Zhao [18] proved the existence and convergence of the global smooth solutions to (1.1). For the multidimensional case, we refer to Zhao [16] and [17]. When $\gamma = 0$ and $h_j = 0$, Chen and Xue [2] proved that global existence and asymptotic behavior of solutions in one space dimension. Later, global existence and optimal decay estimate of solution have been established in [13]. Moreover, they also showed that as time tends to infinity, the global solution approaches the nonlinear diffusion wave described by the self-similar solution of the viscous Burgers equation for $n = 1$. For more study of other type Benjamin-Bona-Mahony-Burgers equations, we may refer to [1], [4], [6], [7], [8] and [15].

Is there time periodic solution to (1.1), which has the same period as $h_j (j = 1, \ldots, n)$? Is the time periodic solution unique and stable? To the best of our knowledge, these are interesting and challenging open problems and few results are available. We shall try to solve these problems in this paper. So our main purpose of this paper is to establish existence, uniqueness and asymptotic stability of time periodic solutions to (1.1). More precisely, existence and uniqueness of time periodic solutions v^{per} are established by decay properties of solutions operator and the contracting mapping principle, provided that the norm of h_j is suitably small. For the details, we refer to Theorem 1. Moreover, the stability of the time periodic solution v^{per} can be studied by investigating the following initial value problem for (1.1) with the initial value

$$t = t_0 : v = v_0(x), \tag{1.2}$$

when the initial data is small perturbation of the time periodic solution for some fixed initial time $t_0 \in \mathbb{R}$. For the details, we refer to Theorem 2.

The study of the global existence and asymptotic behavior of solutions to nonlinear evolution equations has a long history and lots of interesting results have been established. We may refer to [2, 3, 9, 10, 11, 12, 14] and the references therein.

The paper is organized as follows. In Section 2, the decay properties of solution operator to (1.1) are obtained. In Section 3, we prove existence and uniqueness of time periodic solutions to (1.1). Finally, we establish stability of time periodic solutions under suitable conditions in Section 4.

Notations. We give some notations which are used in this paper. Let $\mathcal{F}[u]$ denote the Fourier transform of u defined by

$$\hat{u}(\xi) = \mathcal{F}[u] = \int_{\mathbb{R}^n} e^{-i\xi\cdot x} u(x)dx,$$

and we denote its inverse transform by $\mathcal{F}^{-1}$.

2 Decay property of solution operator

The aim of this section is to establish decay properties of solution operator to the problem (1.1). We first investigate the linear equation of (1.1):

$$v_t - \alpha\Delta v_t - \beta\Delta v + \gamma\Delta^2 v + \sum_{j=1}^{n} f_j'(0)v_{x_j} - g'(0)\Delta v = 0.$$

Taking the Fourier transform, we have

$$(1+\alpha|\xi|^2)\hat{v}_t + \Big[\gamma|\xi|^4 + (\beta+g'(0))|\xi|^2 + i\sum_{j=1}^{n} f_j'(0)\xi_j\Big]\hat{v} = 0. \tag{2.1}$$

The characteristic equation of (2.1) is

$$(1+\alpha|\xi|^2)\lambda + \gamma|\xi|^4 + (\beta+g'(0))|\xi|^2 + i\sum_{j=1}^{n} f_j'(0)\xi_j = 0. \tag{2.2}$$

Let $\lambda(\xi)$ be the corresponding eigenvalues of (2.2), we obtain

$$\lambda(\xi) = \big(-\gamma|\xi|^4 - (\beta+g'(0))|\xi|^2 - i\sum_{j=1}^{n} f_j'(0)\xi_j)\big)/(1+\alpha|\xi|^2).$$

Let

$$\hat{\mathfrak{G}}(\xi,t) = e^{\lambda(\xi)t}. \tag{2.3}$$

We define $\mathfrak{G}(x,t)$ by

$$\mathfrak{G}(x,t) = \mathcal{F}^{-1}[\hat{\mathfrak{G}}(\xi,t)](x), \tag{2.4}$$

where $\mathcal{F}^{-1}$ denotes the inverse Fourier transform.

The decay estimates of the solution operators $\mathfrak{G}(t)$ appearing in the solution formula (2.4) is stated as follows.

Lemma 1. *Let $1 \le p \le 2$, and let k, κ and l be nonnegative integers. Then we have*

$$\|\partial_x^k \mathfrak{G}(t) * \phi\|_{L^2} \le C(1+t)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{2})-\frac{k-\kappa}{2}}\|\partial_x^\kappa\phi\|_{L^p} + Ce^{-ct}\|\partial_x^{k+l}\phi\|_{L^2} \tag{2.5}$$

for $0 \le \kappa \le k$ and $\phi \in W^{\kappa,p}\bigcap H^{k+l}$.

Proof. It follows from the Plancherel theorem and (2.3) that

$$\begin{aligned}\|\partial_x^k \mathfrak{G}(t) * \phi\|_{L^2}^2 &= \int_{\mathbb{R}^n} |\xi|^{2k}|\hat{\mathfrak{G}}(\xi,t)|^2|\hat{\phi}(\xi)|^2 d\xi \\ &\le C\int_{|\xi|\le r_0}|\xi|^{2k}e^{-c|\xi|^2 t}|\hat{\phi}(\xi)|^2 d\xi + Ce^{-ct}\int_{|\xi|\ge r_0}|\xi|^{2k}|\hat{\phi}(\xi)|^2 d\xi =: \mathfrak{I}_1 + \mathfrak{I}_2,\end{aligned} \tag{2.6}$$

where r_0 is a small positive constant. In what follows, we estimate $\mathcal{J}_1$, letting $1/p' + 1/p = 1$, we obtain

$$\mathcal{J}_1 = C\int_{|\xi|\le r_0} \xi|^{2k} e^{-c|\xi|^2 t}|\hat{\phi}(\xi)|^2 d\xi \le C\| \,|\xi|^{\kappa}\hat{\phi}\|^2_{L^{p'}}\Big(\int_{|\xi|\le r_0}|\xi|^{2(k-\kappa)q}e^{-cq|\xi|^2 t}d\xi\Big)^{\frac{1}{q}}$$
$$\le C(1+t)^{-n(\frac{1}{p}-\frac{1}{2})-(k-\kappa)}\|\partial_x^{\kappa}\phi\|^2_{L^p},$$

where we have used the Hölder inequality with $\frac{2}{p'} + \frac{1}{q} = 1$ and the Hausdorff-Young inequality $\|\hat{v}\|_{L^{p'}} \le C\|v\|_{L^p}$ for $v = \partial_x^{\kappa}\phi$. Finally, we can estimate $\mathcal{J}_2$ simply as

$$\begin{aligned}\mathcal{J}_2 &= Ce^{-ct}\int_{|\xi|\ge r_0}|\xi|^{2k}|\hat{\phi}(\xi)|^2 d\xi\\ &\le Ce^{-ct}\int_{|\xi|\ge r_0}|\xi|^{2(k+l)}|\hat{\phi}(\xi)|^2 d\xi \le Ce^{-ct}\|\partial_x^{k+l}\phi\|^2_{L^2}.\end{aligned} \tag{2.7}$$

Combining (2.6) and the above two estimates yields (2.5). Then Lemma 1 is proved. □

As a corollary of Lemma 1, we have the following decay estimate for the term $\mathfrak{S}(t) * (1 - \alpha\Delta)^{-1}\partial_{x_j}F_j$ and $\mathfrak{S}(t) * (1 - \alpha\Delta)^{-1}\Delta G$.

Corollary 1. Let $1 \le p \le 2$, and let k, κ and l be nonnegative integers. Then we have

$$\begin{aligned}\|\partial_x^k\mathfrak{S}(t) * (I-\alpha\Delta)^{-1}\partial_{x_j}F_j\|_{L^2} &\le C(1+t)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{2})-\frac{k+1-\kappa}{2}}\|\partial_x^{\kappa}F_j\|_{L^p}\\ &+ Ce^{-ct}\|\partial_x^{k+l-1}F_j\|_{L^2},\ \forall F_j \in W^{\kappa,p}\bigcap H^{k+l-1}\end{aligned} \tag{2.8}$$

and

$$\begin{aligned}\|\partial_x^k\mathfrak{S}(t) * (I-\alpha\Delta)^{-1}\Delta G\|_{L^2} &\le C(1+t)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{2})-\frac{k+2-\kappa}{2}}\|\partial_x^{\kappa}G\|_{L^p}\\ &+ Ce^{-ct}\|\partial_x^{k+l}G\|_{L^2},\ \forall G \in W^{\kappa,p}\bigcap H^{k+l},\end{aligned} \tag{2.9}$$

where $0 \le \kappa \le k+1$ and $k+l-1 \ge 0$ in (2.8) and $0 \le \kappa \le k+2$ in (2.9).

3 Existence and uniqueness of time periodic solutions

The purpose of this section is to establish existence and uniqueness of time periodic solutions to the problem (1.1), which has the same period as $h_j (j = 1, \dots, n)$. To prove existence and uniqueness of time periodic solutions, we need the following Lemma that has been established in [5] and [19].

Lemma 2. *Assume that $\Phi = \Phi(v)$ is a smooth function satisfying $\Phi(v) = O(|v|^{1+\sigma})$ for $v \to 0$, where $\sigma \ge 1$ is an integer. Let $v \in L^{\infty}$ and $\|v\|_{L^\infty} \le M_0$ for a positive constant M_0. Let $1 \le p,\ q,\ r \le +\infty$ and $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$, and let $k \ge 0$ be an integer. Then we have*

$$\|\partial_x^k\Phi(v)\|_{L^p} \le C\|v\|^{\sigma-1}_{L^\infty}\|v\|_{L^q}\|\partial_x^k v\|_{L^r}.$$

Furthermore, we have

$$\|\partial_x^\alpha(\Phi(v_1)-\Phi(v_2))\|_{L^p}\le C\{(\|\partial_x^\alpha v_1\|_{L^q}+\|\partial_x^\alpha v_2\|_{L^q})\|v_1-v_2\|_{L^r}\\ +(\|v_1\|_{L^r}+\|v_2\|_{L^r})\|\partial_x^\alpha(v_1-v_2)\|_{L^q}\}(\|v_1\|_{L^\infty}+\|v_2\|_{L^\infty})^{\sigma-1},$$

where $C=C(M_0)$ is a constant depending on M_0.

Our existence and uniqueness of time periodic solutions results are stated as follows:

Theorem 1. *Let $n\ge 3$ and $m>n/2$ be integers. For any $j=1,\dots,n$, assume that $h_j\in C([0,T];L^1(\mathbb{R}^n))\bigcap C([0,T];H^{m-1}(\mathbb{R}^n))$ is a periodic function with period T. Put $\mathcal{E}_0=\sum_{j=1}^n\sup_{0\le t\le T}\Big(\|h_j(t)\|_{L^1}+\|h_j(t)\|_{H^{m-1}}\Big)$. Then there exists a positive constant δ_0 such that if $\mathcal{E}_0\le\delta_0$, the problem (1.1) has a unique time periodic solution $v^{per}\in C([0,T];H^m(\mathbb{R}^n))$. Moreover, it holds that*

$$\sup_{0\le t\le T}\|v^{per}(t)\|_{H^m}\le C\mathcal{E}_0.$$

Proof. The proof of Theorem 1 is divided into two steps. The first step is to prove that the solution to the problem (1.1) is periodic solution, provided that there exists a unique solution to the problem (1.1) . The second step is to prove the problem (1.1) admits a unique solution.

Step 1: If there exists a unique solution to the problem (1.1), this unique solution must be time periodic solution. To this end, we define the following integral equation

$$v^{per}(t)=\mathfrak{S}(t-s)*v^{per}(s)+\int_s^t\mathfrak{S}(t-\tau)*(1-\alpha\Delta)^{-1}\\ \times\Big[\sum_{j=1}^n(-F_j(\phi)_{x_j}+h_j(x,t)_{x_j})+\Delta G(\phi)\Big](\tau)d\tau. \tag{3.1}$$

Here $F_j(v)=f_j(v)-f_j(0)-f_j'(0)v=O(v^2)$ $(j=1,\dots,n)$, and $G(v)=g(v)-g(0)-g'(0)v=O(v^2)$. Then (3.1) is the solution to the following problem

$$v_t-\alpha\Delta v_t-(\beta+g'(0))\Delta v+\gamma\Delta^2 v\\ +\sum_{j=1}^n f_j'(0)v_{x_j}+\sum_{j=1}^n F_j(\phi)_{x_j}=\Delta G(\phi)+\sum_{j=1}^n h_j(x,t)_{x_j}$$

with initial value $t=s: v_0=v^{per}(s)$. Choosing $s=-kT$ for $k\in\mathbb{N}$, then (3.1) becomes

$$v^{per}(t)=\mathfrak{S}(t+kT)*v^{per}(s)+\int_{-kT}^t\mathfrak{S}(t-\tau)*(1-\alpha\Delta)^{-1}\\ \times\left[\sum_{j=1}^n(-F_j(\phi)_{x_j}+h_j(x,t)_{x_j})+\Delta G(\phi)\right](\tau)d\tau \tag{3.2}$$

Noting that $n \geq 3$, (2.5) entails that

$$\|\mathfrak{S}(t-s)*\sigma\|_{H^m} \leq C(1+t-s)^{-\frac{n}{4}}(\|\sigma\|_{L^1}+\|\sigma\|_{H^m}) \to 0, \ \ \forall \sigma \in H^m \bigcap L^1 \quad (3.3)$$

as $s \to -\infty$. Since $L^1 \bigcap L^2$ is dense in L^2, (3.3) implies that

$$\|\mathfrak{S}(t-s) * \sigma\|_{H^m} \to 0, \ \ \forall \sigma \in H^m \ \text{ as } s \to -\infty.$$

(2.8) and (2.9) imply that

$$\begin{cases} \|\mathfrak{S}(t-\tau) * (1-\alpha\Delta)^{-1}\partial_{x_j} F_j\|_{H^m} \leq C(1+t-\tau)^{-\frac{n}{4}-\frac{1}{2}}(\|F_j\|_{L^1}+\|F_j\|_{H^m}), \\ \|\mathfrak{S}(t-\tau) * (1-\alpha\Delta)^{-1}\partial_{x_j} h_j\|_{H^m} \leq C(1+t-\tau)^{-\frac{n}{4}-\frac{1}{2}}(\|h_j\|_{L^1}+\|h_j\|_{H^{m-1}}), \\ \|\mathfrak{S}(t-\tau) * (1-\alpha\Delta)^{-1}\Delta G\|_{H^m} \leq C(1+t-\tau)^{-\frac{n}{4}-1}(\|G\|_{L^1}+\|G\|_{H^m}), \end{cases}$$

which together with $\frac{n}{4}+\frac{1}{2} > 1 (n \geq 3)$ entails the converge of the integral in (3.2). Thus, we have

$$v^{per}(t) = \int_{-\infty}^{t} \mathfrak{S}(t-\tau)*(1-\alpha\Delta)^{-1}\Big[\sum_{j=1}^{n}(-F_j(v)_{x_j}+h_j(x,t)_{x_j})+\Delta G(v^{per})\Big](\tau)d\tau. \tag{3.4}$$

Define the mapping

$$\begin{aligned} \mathfrak{M}(v^{per})(t) = \int_{-\infty}^{t} & \mathfrak{S}(t-\tau) * (1-\alpha\Delta)^{-1} \\ & \times \Big[\sum_{j=1}^{n}(-F_j(v)_{x_j} + h_j(x,t)_{x_j}) + \Delta G(v^{per})\Big](\tau)d\tau. \end{aligned} \tag{3.5}$$

Assume that $\mathfrak{M}$ has a unique fixed point, denoted by v_1^{per}, i.e., $\mathfrak{M}(v_1^{per}) = v_1^{per}$. Let $v_2^{per}(t) = v_1^{per}(t+T)$, then it follows from (3.5) and $h_j(x,\tau+T) = h_j(x,\tau)(j=1,\ldots,n)$ that

$$\begin{aligned} v_2^{per}(t) &= v_1^{per}(t+T) = \mathfrak{M}(v_1^{per}(t+T)) = \int_{-\infty}^{t+T} \mathfrak{S}(t+T-\tau) \\ & * (1-\alpha\Delta)^{-1}\Big[\sum_{j=1}^{n}(-F_j(v_1^{per})_{x_j} + h_j(x,t)_{x_j}) + \Delta G(v_1^{per})\Big](\tau)d\tau \\ &= \int_{-\infty}^{t} \mathfrak{S}(t+T-(\tau+T)) * (1-\alpha\Delta)^{-1}\Big[\sum_{j=1}^{n}(-F_j(v_1^{per})_{x_j} \\ & + h_j(x,t)_{x_j}) + \Delta G(v_1^{per})\Big](\tau+T)d\tau = \int_{-\infty}^{t} \mathfrak{S}(t-\tau) * (1-\alpha\Delta)^{-1} \\ & \times \Big[\sum_{j=1}^{n}(-F_j(v_2^{per})_{x_j} + h_j(x,t)_{x_j}) + \Delta G(v_2^{per})\Big](\tau)d\tau = \mathfrak{M}(v_2^{per}(t)). \end{aligned}$$

Then $v_2^{per}(t)$ is also fixed point. Due to uniqueness of the fixed point, we have

$$v_1^{per}(t) = v_2^{per}(t) = v_1^{per}(t+T),$$

which implies $v_1^{per}(t)$ is a periodic function with period T. Therefore, the fixed point of the mapping $\mathfrak{M}$ is a periodic function with period T, provided that the mapping $\mathfrak{M}$ has a unique fixed point.

Step 2: The problem (1.1) admits a unique solution.

In this step, we shall prove that existence of solutions to the problem (1.1) in the function space $C([0,T];H^m(\mathbb{R}^n))$ by the contraction mapping theorem. To this end, define the function space

$$X = \{v^{per} \in C([0,T];H^m(\mathbb{R}^n)) : \|v^{per}\|_X < \infty\},$$

where $\|v^{per}\|_X = \sup_{0\le t\le T} \|v^{per}(t)\|_{H^m}$. For $R > 0$, we define

$$Y = \{v^{per} \in X : \|v^{per}\|_X \le R\}.$$

To prove that there exists a unique solution to the problem (1.1), it is suffice to prove that $\mathfrak{M}$ has a unique fixed point in the function space Y. For $\forall v^{per} \in Y$ and $0 \le k \le m$, it follows from (3.5) and Minkowski inequality that,

$$\begin{aligned}
\|\partial_x^k \mathfrak{M}(v^{per})(t)\|_{L^2} &\le \int_{-\infty}^t \|\partial_x^k \mathfrak{G}(t-\alpha\tau) * (1-\Delta)^{-1} \sum_{j=1}^n -F_j(v^{per})_{x_j}(\tau)\|_{L^2} d\tau \\
&+ \int_{-\infty}^t \|\partial_x^k \mathfrak{G}(t-\alpha\tau) * (1-\Delta)^{-1} \sum_{j=1}^n h_j(x,t)_{x_j}(\tau)\|_{L^2} d\tau \\
&+ \int_{-\infty}^t \|\partial_x^k \mathfrak{G}(t-\alpha\tau) * (1-\Delta)^{-1} \Delta G(v^{per})(\tau)(\tau)\|_{L^2} d\tau =: K_1 + K_2 + K_3.
\end{aligned}$$

(2.8), (2.9), Lemma 2 and Sobolev embedding theorem entails that

$$\begin{aligned}
K_1 &\le C \int_{-\infty}^t (1+t-\tau)^{-\frac{n}{4}-\frac{k+1}{2}} \sum_{j=1}^n \|F_j(v^{per})(\tau)\|_{L^1} d\tau \\
&\quad + C \int_{-\infty}^t e^{-c(t-\tau)} \sum_{j=1}^n \|\partial_x^k F_j(v^{per})(\tau)\|_{L^2} d\tau \\
&\le C \int_{-\infty}^t (1+t-\tau)^{-\frac{n}{4}-\frac{k+1}{2}} \left(\|v^{per}(\tau)\|_{L^2}^2 + \|v^{per}(\tau)\|_{L^\infty} \|\partial_x^k v^{per}(\tau)\|_{L^2}\right) d\tau \\
&\le C\|v^{per}\|_X^2 \int_{-\infty}^t (1+t-\tau)^{-\frac{n}{4}-\frac{k+1}{2}} d\tau \le C\|v^{per}\|_X^2, \\
K_2 &\le C \int_{-\infty}^t (1+t-\tau)^{-\frac{n}{4}-\frac{k+1}{2}} \sum_{j=1}^n \|h_j(\tau)\|_{L^1} d\tau \\
&\quad + C \int_{-\infty}^t e^{-c(t-\tau)} \sum_{j=1}^n \|\partial_x^{(k-1)_+} h_j(\tau)\|_{L^2} d\tau \\
&\le C \int_{-\infty}^t (1+t-\tau)^{-\frac{n}{4}-\frac{k+1}{2}} \sum_{j=1}^n \Big(\|h_j(\tau)\|_{L^1} + \|\partial_x^{(k-1)_+} h_j(\tau)\|_{L^2}\Big) d\tau
\end{aligned}$$

$$
\begin{aligned}
&\leq C \sup_{0\leq t\leq T} \sum_{j=1}^{n} \Big(\|h_j(t)\|_{L^1} + \|h_j(t)\|_{H^{m-1}} \Big) \int_{-\infty}^{t} (1+t-\tau)^{-\frac{n}{4}-\frac{k+1}{2}} d\tau \\
&\leq C \sup_{0\leq t\leq T} \sum_{j=1}^{n} \Big(\|h_j(t)\|_{L^1} + \|h_j(t)\|_{H^{m-1}} \Big),
\end{aligned}
$$

$$
\begin{aligned}
K_3 &\leq C \int_{-\infty}^{t} (1+t-\tau)^{-\frac{n}{4}-\frac{k+2}{2}} \|G(v^{per})(\tau)\|_{L^1} d\tau \\
&\quad + C \int_{-\infty}^{t} e^{-c(t-\tau)} \|\partial_x^k G(v^{per})(\tau)\|_{L^2} d\tau \\
&\leq C \int_{-\infty}^{t} (1+t-\tau)^{-\frac{n}{4}-\frac{k+2}{2}} \left(\|v^{per}(\tau)\|_{L^2}^2 + \|v^{per}(\tau)\|_{L^\infty} \|\partial_x^k v^{per}(\tau)\|_{L^2} \right) d\tau \\
&\leq C \|v^{per}\|_X^2 \int_{-\infty}^{t} (1+t-\tau)^{-\frac{n}{4}-\frac{k+2}{2}} d\tau \leq C \|v^{per}\|_X^2,
\end{aligned}
$$

where $(k-1)_+ = \max\{0, k-1\}$.
Combining above three estimates gives

$$
\|\partial_x^k \mathfrak{M}(v^{per})(t)\|_{L^2} \leq C \|v^{per}\|_X^2 + C \sup_{0\leq t\leq T} \sum_{j=1}^{n} \Big(\|h_j(t)\|_{L^1} + \|h_j(t)\|_{H^{m-1}} \Big),
$$

which implies

$$
\|\mathfrak{M}(v^{per})\|_X \leq C \|v^{per}\|_X^2 + C\mathcal{E}_0.
$$

Taking $R = 4C\mathcal{E}_0$ and letting $\mathcal{E}_0$ be suitably small, we have

$$
\|\mathfrak{M}(v^{per})\|_X \leq 2C\mathcal{E}_0 \leq R. \tag{3.6}
$$

Finally, we prove $\mathfrak{M}$ is a strictly contracting mapping. $\forall \tilde{v}^{per}, \bar{v}^{per} \in Y$, owing to (3.5), it holds that

$$
\begin{aligned}
\mathfrak{M}(\tilde{v}^{per}) - \mathfrak{M}(\bar{v}^{per})(t) &= \int_{-\infty}^{t} \mathfrak{G}(t-\tau) * (1-\alpha\Delta)^{-1} \\
&\times \Big\{ \sum_{j=1}^{n} \partial_{x_j} [F_j(\bar{v}^{per}) - F_j(\tilde{v}^{per})] + \Delta[G(\tilde{v}^{per}) - G(\bar{v}^{per})] \Big\} (\tau) d\tau.
\end{aligned} \tag{3.7}
$$

By (3.7), Minkowski inequality, (2.8), (2.9), Lemma 2 and Sobolev embedding theorem, we obtain

$$
\begin{aligned}
&\|\partial_x^k (\mathfrak{M}(\tilde{v}^{per}) - \mathfrak{M}(\bar{v}^{per}))(t)\|_{L^2} \leq C \int_{-\infty}^{t} (1+t-\tau)^{-\frac{n}{4}-\frac{k+1}{2}} \\
&\quad \times \Big\{ \sum_{j=1}^{n} \|\big(F_j(\tilde{v}^{per}) - F_j(\bar{v}^{per})\big)(\tau)\|_{L^1} + \|\big(G(\tilde{v}^{per}) - G(\bar{v}^{per})\big)(\tau)\|_{L^1} \Big\} d\tau \\
&\quad + C \int_{-\infty}^{t} e^{-c(t-\tau)} \Big\{ \sum_{j=1}^{n} \|\partial_x^k (F_j(\tilde{v}^{per}) - F_j(\bar{v}^{per}))(\tau)\|_{L^2}
\end{aligned}
$$

$$
\begin{aligned}
&+\|\partial_x^k(G(\tilde{v}^{per})-G(\bar{v}^{per}))(\tau)\|_{L^2}\Big\}d\tau \\
&\leq C\int_{-\infty}^{t}(1+t-\tau)^{-\frac{n}{4}-\frac{k+1}{2}}\Big\{\sum_{j=1}^{n}\|\big(F_j(\tilde{v}^{per})-F_j(\bar{v}^{per})\big)(\tau)\|_{L^1} \\
&+\|\big(G(\tilde{v}^{per})-G(\bar{v}^{per})\big)(\tau)\|_{L^1}+\sum_{j=1}^{n}\|\partial_x^k(F_j(\tilde{v}^{per})-F_j(\bar{v}^{per}))(\tau)\|_{L^2} \\
&+\|\partial_x^k(G(\tilde{v}^{per})-G(\bar{v}^{per}))(\tau)\|_{L^2}\Big\}d\tau \\
&\leq C\int_{-\infty}^{t}(1+t-\tau)^{-\frac{n}{4}-\frac{k+1}{2}}\Big\{(\|(\tilde{v}^{per}(\tau)\|_{L^2}+\|\bar{v}^{per}(\tau)\|_{L^2})\|(\tilde{v}^{per}-\bar{v}^{per})(\tau)\|_{L^2} \\
&+\|(\tilde{v}^{per}-\bar{v}^{per})(\tau)\|_{L^\infty}(\|\partial_x^k\tilde{v}^{per}(\tau)\|_{L^2}+\|\partial_x^k\bar{v}^{per}(\tau)\|_{L^2}) \\
&+(\|\tilde{v}^{per}(\tau)\|_{L^\infty}+\|\bar{v}^{per}(\tau)\|_{L^\infty})\|\partial_x^k(\tilde{v}^{per}-\bar{v}^{per})(\tau)\|_{L^2}\Big\}d\tau \\
&\leq C(\|\tilde{v}^{per}\|_X+\|\bar{v}^{per}\|_X)\|\tilde{v}^{per}-\bar{v}^{per}\|_X\int_{-\infty}^{t}(1+t-\tau)^{-\frac{n}{4}-\frac{k+1}{2}}d\tau \\
&\leq C(\|\tilde{v}^{per}\|_X+\|\bar{v}^{per}\|_X)\|\tilde{v}^{per}-\bar{v}^{per}\|_X,\quad \forall 0\leq k\leq m,
\end{aligned}
$$

which implies

$$
\|(\mathfrak{M}(\tilde{v}^{per})-\mathfrak{M}(\bar{v}^{per}))(t)\|_X\leq CR\|\tilde{v}^{per}-\bar{v}^{per}\|_X. \tag{3.8}
$$

Recalling that $R=4C\mathcal{E}_0$ and letting $\mathcal{E}_0$ suitably small and combining (3.8) yields

$$
\|(\mathfrak{M}(\tilde{v}^{per})-\mathfrak{M}(\bar{v}^{per}))(t)\|_X\leq \frac{1}{2}\|\tilde{v}^{per}-\bar{v}^{per}\|_X. \tag{3.9}
$$

(3.6) and (3.9) imply that $\mathfrak{M}$ is a strictly contracting mapping. Consequently, we conclude that there exists a unique fixed point $v^{per}\in Y$ of the mapping $\mathfrak{M}$, which is a unique solution to (1.1).

Step 1 and Step 2 entails that the problem (1.1) exists a unique time periodic solution. We have complete the proof of Theorem 1. □

Remark 1. Due to the integral with respect to time, in this paper, we only prove that existence and uniqueness of time periodic solutions when the space dimension $n\geq 3$. We shall discuss existence and uniqueness of time periodic solutions for $n=1,2$.

4 Stability of time periodic solutions

In this section, we shall prove the stability of time periodic solutions established in Theorem 1. The asymptotic stability of time periodic solutions is stated as follows.

Theorem 2. *Assume the conditions of Theorem 1 hold and* $v_0\in H^m\bigcap L^1$. *Put*

$$
\mathcal{E}_1=\|v_0-v^{per}(t_0)\|_{L^1}+\|v_0-v^{per}(t_0)\|_{H^m}.
$$

Then there exists a positive constant δ_1 *such that if* $\mathcal{E}_1\leq\delta_1$, *the problem* (1.1), (1.2) *has a unique global solution* $v\in C([t_0,\infty);H^m(\mathbb{R}^n))$. *Moreover,*

$$
\|(v-v^{per})(t)\|_{L^2}\leq C\mathcal{E}_1(1+t)^{-\frac{n}{4}}. \tag{4.1}
$$

Remark 2. Notice that for time periodic force $\partial_{x_j} h_j(x,t)(j=1,\dots,n)$, there is no time decay in the forcing term so that the large time behavior is more complicated. The method used in this paper relies on the convergence of the integral defined in (3.4). So this convergence can be proved when the space dimension $n \geq 3$. Therefore, it is still an open problem for $n \leq 2$.

Proof. Firstly, we shall prove the global existence of the solution to the problem (1.1)–(1.2). Without loss of generality, we can assume $t_0 = 0$. Let v^{per} be the periodic solution constructed in Theorem 1 and let v be a solution to the initial value problem (1.1)–(1.2). Then $V = v - v^{per}$ satisfies the following initial value problem

$$V_t - \alpha \Delta V_t - (\beta + g'(0))\Delta V + \gamma \Delta^2 V + \sum_{j=1}^{n} f_j'(0) V_{x_j}$$

$$+ \sum_{j=1}^{n} (F_j(v) - F_j(v^{per}))_{x_j} = \Delta(G(v) - G(v^{per})), \tag{4.2}$$

$$t = 0: \quad V = v_0(x) - v^{per}(x,0). \tag{4.3}$$

The existence and uniqueness of local solutions may be established by the contraction mapping principle (cf. [3, 12, 14]). In what follow, global existence of solutions to the problem (1.1)–(1.2) will be proved by continuous argument. To this end, we assume that

$$\mathcal{M} = \sup_{0 \leq t < T_0} \|V(t)\|_{H^m} \leq 2C_0 \mathcal{E}_1.$$

where T_0 is the maximal time of existence of local solutions.
We may transform the problem (4.2)–(4.3) into the following integral equation

$$V(t) = \mathfrak{S}(t) * (v_0(x) - v^{per}(0)) + \int_0^t \mathfrak{S}(t-\tau) * (1 - \alpha\Delta)^{-1} \tag{4.4}$$

$$\times \Big\{ \sum_{j=1}^{n} (F_j(v^{per}) - F_j(V + v^{per}))_{x_j} + \Delta[G(V + v^{per}) - G(v^{per})] \Big\}(\tau) d\tau.$$

Equation (4.4) and the Minkowski inequality entails that

$$\|\partial_x^k V(t)\|_{L^2} \leq \|\partial_x^k \mathfrak{S}(t) * (v_0(x) - v^{per}(0))\|_{L^2} + \sum_{j=1}^{n} \int_0^t \|\partial_x^k \mathfrak{S}(t-\tau) * (1-\alpha\Delta)^{-1}$$

$$\times \partial_{x_j}[F_j(v^{per}) - F_j(V + v^{per})](\tau)\|_{L^2} d\tau + \int_0^t \|\partial_x^k \mathfrak{S}(t-\tau) * (1-\alpha\Delta)^{-1}$$

$$\times \Delta[G(V + v^{per}) - G(v^{per})](\tau)\|_{L^2} d\tau =: I_1 + I_2 + I_3. \tag{4.5}$$

Making use of (2.5), we have

$$\begin{aligned} I_1 \leq & C(1+t)^{-\frac{n}{4}-\frac{k}{2}} \|v_0(x) - v^{per}(0)\|_{L^1} + Ce^{-ct} \|\partial_x^k (v_0(x) - v^{per}(0))\|_{L^2} \\ \leq & C(1+t)^{-\frac{n}{4}-\frac{k}{2}} \mathcal{E}_1. \end{aligned} \tag{4.6}$$

It follows from (2.8), Lemma 2 and Sobolev embedding theorem, Theorem 1 that

$$
\begin{aligned}
I_2 \leq & C\sum_{j=1}^{n}\int_0^t (1+t-\tau)^{-\frac{n}{4}-\frac{k+1}{2}}\|(F_j(V+v^{per})-F_j(v^{per}))(\tau)\|_{L^1}d\tau \\
& + C\sum_{j=1}^{n}\int_0^t e^{-c(t-\tau)}\|\partial_x^k(F_j(V+v^{per})-F_j(v^{per}))(\tau)\|_{L^2}d\tau \\
\leq & C\int_0^t (1+t-\tau)^{-\frac{n}{4}-\frac{k+1}{2}}\|V(\tau)\|_{L^2}(\|V(\tau)\|_{L^2}+\|v^{per}(\tau)\|_{L^2})d\tau \\
& + C\int_0^t e^{-c(t-\tau)}\Big\{(\|V\|_{L^\infty}+\|v^{per}\|_{L^\infty})\|\partial_x^k V(\tau)\|_{L^2} \\
& + \|V(\tau)\|_{L^\infty}(\|\partial_x^k V\|_{L^2}+\|\partial_x^k v^{per}\|_{L^2})\Big\}d\tau \\
\leq & C(\mathcal{M}^2+\delta_0\mathcal{M})\int_0^t (1+t-\tau)^{-\frac{n}{4}-\frac{k+1}{2}}d\tau + C(\mathcal{M}^2+\delta_0\mathcal{M})\int_0^t e^{-c(t-\tau)}d\tau \\
\leq & C\Big(\mathcal{M}^2+\delta_0\mathcal{M}\Big).
\end{aligned} \tag{4.7}
$$

Using (2.9) and the same procedure leading to (4.7), it holds that

$$
\begin{aligned}
I_3 \leq & C\int_0^t (1+t-\tau)^{-\frac{n}{4}-\frac{k+2}{2}}\|(G(V+v^{per})-G(v^{per}))(\tau)\|_{L^1}d\tau \\
& + C\int_0^t e^{-c(t-\tau)}\|\partial_x^k(G(V+v^{per})-G(v^{per}))(\tau)\|_{L^2}d\tau \\
\leq & C\int_0^t (1+t-\tau)^{-\frac{n}{4}-\frac{k+2}{2}}\|V(\tau)\|_{L^2}(\|V(\tau)\|_{L^2}+\|v^{per}(\tau)\|_{L^2})d\tau \\
& + C\int_0^t e^{-c(t-\tau)}\Big\{(\|V\|_{L^\infty}+\|v^{per}\|_{L^\infty})\|\partial_x^k V(\tau)\|_{L^2} \\
& + \|V(\tau)\|_{L^\infty}(\|\partial_x^k V\|_{L^2}+\|\partial_x^k v^{per}\|_{L^2})\Big\}d\tau \\
\leq & C(\mathcal{M}^2+\delta_0\mathcal{M})\int_0^t (1+t-\tau)^{-\frac{n}{4}-\frac{k+2}{2}}d\tau + C(\mathcal{M}^2+\delta_0\mathcal{M})\int_0^t e^{-c(t-\tau)}d\tau \\
\leq & C\Big(\mathcal{M}^2+\delta_0\mathcal{M}\Big).
\end{aligned} \tag{4.8}
$$

We insert (4.6)–(4.8) into (4.5) and obtain

$$
\mathcal{M} \leq C_1\mathcal{E}_1 + C\mathcal{M}^2 + \delta_0\mathcal{M}. \tag{4.9}
$$

Letting $C_0 = 4C_1$, (4.9) implies that $\mathcal{M} \leq C_0\mathcal{E}_1$, provided that δ_0 and $\mathcal{E}_1$ are suitably small. By standard continuous argument, we conclude that the problem (4.2)–(4.3) admits a unique global solution V. Therefore, the problem (1.1)–(1.2) admits a unique global solution v.

Next we shall prove (4.1). We introduce the quantity

$$
\mathcal{E}(t) = \sup_{0\leq\tau\leq t}(1+\tau)^{\frac{n}{4}}\|V(\tau)\|_{L^2}.
$$

To prove (4.1), it is suffice to prove that $\mathcal{E}(t) \le C\mathcal{E}_1$. Equation (4.5) with $k=0$ gives

$$\|V(t)\|_{L^2} \le: J_1 + J_2 + J_3. \tag{4.10}$$

Equation (4.6) implies

$$J_1 \le C(1+t)^{-\frac{n}{4}}\mathcal{E}_1. \tag{4.11}$$

(2.8), Lemma 2 and Sobolev embedding theorem, Theorem 1 entails that

$$\begin{aligned}
J_2 \le\ & C\int_0^t (1+t-\tau)^{-\frac{n}{4}-\frac{1}{2}}\|(F_j(V+v^{per})-F_j(v^{per}))(\tau)\|_{L^1}d\tau \\
& +C\int_0^t e^{-c(t-\tau)}\|(F_j(V+v^{per})-F_j(v^{per}))(\tau)\|_{L^2}d\tau \\
\le\ & C\int_0^t (1+t-\tau)^{-\frac{n}{4}-\frac{1}{2}}\|V(\tau)\|_{L^2}(\|V(\tau)\|_{L^2}+\|v^{per}(\tau)\|_{L^2})d\tau \\
& +C\int_0^t e^{-c(t-\tau)}(\|V\|_{L^\infty}\|V(\tau)\|_{L^2}+\|v^{per}(\tau)\|_{L^\infty}\|V(\tau)\|_{L^2})d\tau \\
\le\ & C\mathcal{E}^2(t)\int_0^t (1+t-\tau)^{-\frac{n}{4}-\frac{1}{2}}(1+\tau)^{-\frac{n}{2}}d\tau \\
& +C\delta_0\mathcal{E}(t)\int_0^t (1+t-\tau)^{-\frac{n}{4}-\frac{1}{2}}(1+\tau)^{-\frac{n}{4}}d\tau \\
& +C(\delta_0+\mathcal{E}_1)\mathcal{E}(t)\int_0^t e^{-c(t-\tau)}(1+\tau)^{-\frac{n}{4}}d\tau \\
\le\ & C\Big(\mathcal{E}^2(t)+\delta_0\mathcal{E}(t)+\mathcal{E}_1\mathcal{E}(t)\Big)(1+t)^{-\frac{n}{4}}.
\end{aligned}$$

Similarly, we have from (2.9)

$$\begin{aligned}
J_3 \le\ & C\int_0^t (1+t-\tau)^{-\frac{n}{4}-1}\|(G(V+v^{per})-G(v^{per}))(\tau)\|_{L^1}d\tau \\
& + C\int_0^t e^{-c(t-\tau)}\|(G(V+v^{per})-G(v^{per}))(\tau)\|_{L^2}d\tau \\
\le\ & C\int_0^t (1+t-\tau)^{-\frac{n}{4}-\frac{1}{2}}\|V(\tau)\|_{L^2}(\|V(\tau)\|_{L^2}+\|v^{per}(\tau)\|_{L^2})d\tau \\
& + C\int_0^t e^{-c(t-\tau)}(\|V\|_{L^\infty}\|V(\tau)\|_{L^2}+\|v^{per}(\tau)\|_{L^\infty}\|V(\tau)\|_{L^2})d\tau \\
\le\ & C\mathcal{E}^2(t)\int_0^t (1+t-\tau)^{-\frac{n}{4}-1}(1+\tau)^{-\frac{n}{2}}d\tau \\
& + C\delta_0\mathcal{E}(t)\int_0^t (1+t-\tau)^{-\frac{n}{4}-1}(1+\tau)^{-\frac{n}{4}}d\tau + C(\delta_0+\mathcal{E}_1)\mathcal{E}(t) \\
& \times\int_0^t e^{-c(t-\tau)}(1+\tau)^{-\frac{n}{4}}d\tau \le C\Big(\mathcal{E}^2(t)+\delta_0\mathcal{E}(t)+\mathcal{E}_1\mathcal{E}(t)\Big)(1+t)^{-\frac{n}{4}}.
\end{aligned} \tag{4.12}$$

Inserting (4.11)–(4.12) into (4.10) yields

$$\mathcal{E}(t) \le C\mathcal{E}_1 + C\mathcal{E}^2(t) + \delta_0\mathcal{E}(t) + \mathcal{E}_1\mathcal{E}(t).$$

Therefore, we arrive at

$$\mathcal{E}(t) \leq C\mathcal{E}_1,$$

provided that δ_0 and $\mathcal{E}_1$ are suitably small. Theorem 2 is proved. □

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